

Instrument choice in pre-contest investment

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Abstract

We study a two-period contest in which two risk-neutral rivals can invest before a subsequent all-pay auction in order to alter the playing field. Pre-contest investments can take the form of a head start, which shifts a player's score additively, or a bias, which raises the productivity of effort multiplicatively. We characterize the subgame-perfect equilibria of the game. In any pure-strategy equilibrium, at most one player invests, and the investor uses at most one instrument. The choice between head start and bias depends only on the rival's valuation: against sufficiently weak rivals, the investor chooses a head start, whereas higher rival valuations induce a switch to bias. Equilibrium existence requires that the chosen investment make both head-start and bias deviations by the rival unprofitable, and we characterize the corresponding thresholds. In a quadratic-cost example, the model generates both reinforcement equilibria, in which the high-value player invests, and preemptive equilibria, in which the low-value player invests. More generally, the analysis shows that allowing multiple investment instruments can fundamentally alter equilibrium existence: the monotonicity property familiar from one-instrument environments need not hold once contestants can choose between distinct forms of strategic advantage. **Keywords:** all-pay auction, pre-contest investment, additive head start, multiplicative bias

1 Introduction

Contest theory is used to model competition over a prize in which rivals' bids or efforts are irretrievable and sunk. The theory finds wide application in the areas of

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economics, political science, management, and sports.¹

Many contests are preceded by strategic preparation. Before competing for a prize, rivals can invest in ways that improve their position in the ensuing competition. Such investments may create an *additive* advantage, by giving a contestant a head start that improves winning chances independently of contest effort, or a *multiplicative* advantage (bias), by making effort more productive at the contest stage. We study how the availability of these two forms of pre-contest investment affects equilibrium behavior in contests.

To do so, we develop a two-period model in which two contestants first invest and then compete in an all-pay auction. In the contest stage, the winner is the player with the higher score. A player's score depends on contest effort, but may also reflect two additional advantages accumulated in the investment stage: a head start and a bias. In the first period, each player can invest in head start, in bias, or in both. We ask whether players invest before the contest, which instrument they choose, and how much they invest. We focus on a setting in which contestants differ in prize valuations but share the same strictly convex investment cost function, and where investment in head start and bias draws on a common resource such as time, managerial attention, or budget. This allows us to isolate the strategic roles of additive and multiplicative advantages rather than differences in investment technology.

Three main results emerge. First, in any pure-strategy subgame-perfect equilibrium, at most one contestant invests, and the investing player uses at most one instrument. Second, the investor's choice between head start and bias is governed entirely by the rival's valuation: sufficiently low rival valuations induce head-start investment, while higher rival valuations eventually induce bias. Third, equilibrium existence depends on whether the investor's choice makes both head-start and bias deviations by the rival unprofitable. When this condition fails, no pure-strategy equilibrium exists. In a quadratic-cost example, these forces generate both *reinforcement* equilibria, in which the high-value player invests, and *preemptive* equilibria, in which the low-value player invests.

A broader implication of the analysis is that expanding the set of strategic instruments changes equilibrium behavior fundamentally. In one-instrument environments, equilibrium existence follows a natural monotonicity logic: if the low-value player can profitably invest and leave the rival with no profitable deviation, then the high-value

¹Konrad (2009) is an excellent introduction to contest models and their application.

player can do so as well. In our setting, this implication need not hold. Once players can choose between two distinct instruments, equilibrium existence depends not only on how much they invest, but also on the form that investment takes. The low-value player may therefore be able to sustain the equilibrium by selecting the instrument that is particularly effective at making the stronger rival's preferred deviation unprofitable, even though the reverse need not be true. The availability of multiple instruments thus transforms the pre-contest investment problem from one of pure strength into one of strategic design.

The model applies to a variety of economic environments in which contestants prepare for later competition by investing either in infrastructure or in capability. In sales contests, for example, firms may invest in installed systems, customer pipelines, or marketing assets that act as a head start, or in training that raises the effectiveness of later selling effort. In R&D races, contestants may invest either in equipment and testing capacity that move them directly closer to the goal, or in specialized human capital that increases the productivity of research effort. Similar distinctions arise in procurement, consulting tenders, and other competitive selection processes in which both initial positioning and the productivity of subsequent effort matter for success.

We relate to three strands of literature. First, there is a large literature on contest design in which a principal manipulates contestants' positions through head starts or biases before the contest begins. This includes work on affirmative action, favoritism, and optimal contest design, such as Fu (2006), Li and Yu (2012), Franke et al. (2018), and Chowdhury et al. (2023). In these papers, additive and multiplicative advantages are instruments chosen (at no cost) by a designer to pursue objectives such as revenue, effort, or diversity. By contrast, we study a decentralized environment in which contestants themselves choose whether and how to create such costly advantages.

Second, our paper contributes to the literature on endogenous pre-contest investment. Existing work considers investments that improve the productivity of effort, build human capital, reduce the cost of effort, or otherwise alter the contest technology; see, among others, Keskin and Sağlam (2019), Schaller and Skaperdas (2020), Clark and Kundu (2026), Grossmann and Dietl (2009), Grossmann et al. (2011), Münster (2007a), and Fu and Lu (2009). Our contribution differs from this literature by allowing contestants to choose between two distinct forms of strategic advantage within a common investment framework and by characterizing when each instrument is used in equilibrium.

Third, the paper is closest to analyses of all-pay contests in which pre-contest investment endogenously generates a single advantage, either an additive head start or a multiplicative bias (Clark et al., 2025, 2026). Those environments deliver the monotonicity property described above: if equilibrium exists with investment by the low-value player, then equilibrium also exists with investment by the high-value player. We show that this property need not survive once contestants can choose between multiple instruments. Even though only one instrument is used in equilibrium, the investor must choose an action that leaves the opponent with no profitable deviation along either margin, and this changes conditions for equilibrium existence in a fundamental way.

The remainder of the paper is organized as follows. Sections 2 and 3 present the model and characterize equilibrium. Section 4 illustrates the results under quadratic investment costs, which permit closed-form solutions. Section 5 shows how multiple investment instruments alter equilibrium existence relative to the one-instrument benchmark. Section 6 concludes. All proofs are collected in the Appendix.

2 The model

Two risk-neutral rivals play a two-period game. In the first period, players invest, and in the second, they compete in a contest. Specifically, in period 1, player $i \in \{1, 2\}$ can make an investment $y_i \geq 0$ to gain an advantage in the upcoming contest. Investment is costly, and the investment cost is given by an increasing and convex function $c(y_i)$ with $c(0) = c'(0) = 0; c'(y) > 0, y > 0; c''(y) > 0$. A player can gain an advantage in two ways: (i) by securing a head start in the contest, and (ii) by increasing the effectiveness of his contest effort, which we will term bias. Let $h_i \geq 0$ and $b_i \geq 0$ denote the respective investments in a head start and bias, such that $h_i + b_i = y_i$ and total investment cost is $c(h_i + b_i)$. By making head start and bias identical contributors to a player's investment costs, our aim is to focus on the different roles the two instruments play in the second-period contest as described below. This cost structure arises naturally when the investment opportunities draw from a common resource such as time or a budget.²

²The multi-tasking models of among others Holmström and Milgrom (1991) and Fehr and Schmidt (2004) use the same assumption. Münster (2007b) uses a convex cost for the sum of an agent's sabotage activities in a contest, and Keskin and Sağlam (2022) use a similar function in a dynamic contest where fatigue depends on cumulative effort.

In period 2, the two rivals play an all-pay auction to win a prize which they value at V_1 and V_2 , respectively, such that $V_1 \geq V_2 > 0$. The win probability of player i is determined by his score (S_i) in the period-2 contest, which depends on contest effort (x_i) and investments in period 1:³

$$S_i = h_i + (1 + b_i) x_i.$$

Contest effort has a unit cost for each player. Player i has an expected (gross) payoff from taking part in the period-2 contest of

$$Eu_i = \rho_i V_i - x_i,$$

where ρ_i is that player's probability of winning the contest, given by:

$$\rho_i(S_1, S_2) = \begin{cases} 1, & \text{if } S_i > S_j; \\ \frac{1}{2}, & \text{if } S_i = S_j; \text{ and} \\ 0, & \text{if } S_i < S_j. \end{cases}$$

The probability of player $j \neq i$ winning is $1 - \rho_i(S_1, S_2)$. Player i 's expected net payoff in this game is then $Eu_i - c(y_i)$.

Observe that

$$S_1 \geq S_2 \Leftrightarrow \frac{h_1 - h_2}{1 + b_1} + x_1 \geq \frac{1 + b_2}{1 + b_1} x_2,$$

which shows that the net advantage can be represented by both an additive component and a multiplicative component. If $h_1 > h_2$, then player 1 has an additive advantage, and if $b_1 > b_2$, then player 1 has a multiplicative advantage.⁴

3 Equilibrium analysis

We look for a subgame-perfect Nash equilibrium of the game in which the players first make investment choices, and then compete in the contest. We first solve the second-period contest for given (h, b) (Lemma 1), then characterize optimal first-period in-

³The constant 1 is baseline effort productivity in the absence of bias. This baseline value naturally generates the threshold $V_i = 1$ that appears repeatedly in the investment characterization.

⁴In the terminology of Konrad (2002) and Kirkegaard (2012), such an additive advantage is a head start for player 1, while the multiplicative advantage for 1 is a handicap for player 2.

vestments (Lemmas 2–8), and finally combine these to characterize the subgame-perfect Nash equilibrium (Proposition 1).

3.1 The second-period contest

When the contest starts, investments have taken place which determine the starting point for each competitor at this stage. Define

$$\eta_i := h_j + (1 + b_j)V_j, i, j \in \{1, 2\}, i \neq j, \quad (1)$$

as the hurdle for player i induced by player j when the latter invests in a head start h_j and a bias b_j . Essentially, η_i summarizes the impact that the rival has on player i 's starting point in the contest. The head start of player j shifts the bar for player i additively, whereas the bias of player j tilts the playing field multiplicatively.

Lemma 1 characterizes the unique mixed-strategy equilibrium of the second-period all-pay auction for given investments (h, b) .⁵

Lemma 1. *Let $i, j \in \{1, 2\}, i \neq j$, and assume $(1 + b_i)V_i \geq (1 + b_j)V_j$.*

1. *If $h_i \geq \eta_i$, then $x_i = x_j = 0$; $Eu_i = V_i$, and $Eu_j = 0$.*
2. *If $h_j \leq h_i < \eta_i$, then the unique mixed-strategy equilibrium in the period-2 contest has expected efforts and payoffs given by*

$$\begin{aligned} Ex_i &= \frac{(\eta_i - h_i)^2}{2(1 + b_i)(\eta_i - h_j)}; & Eu_i &= \frac{\eta_j - \eta_i}{1 + b_i}; \\ Ex_j &= \frac{(\eta_i - h_i)(\eta_i + h_i - 2h_j)}{2(1 + b_j)(\eta_j - h_i)}; & Eu_j &= 0. \end{aligned}$$

3. *If $\eta_i < \eta_j$ and $h_i < h_j$, then the unique mixed-strategy equilibrium in the period-2 contest has expected efforts and payoffs given by*

$$\begin{aligned} Ex_i &= \frac{\eta_i - 2h_i + h_j}{2(1 + b_i)}; & Eu_i &= \frac{\eta_j - \eta_i}{1 + b_i}; \\ Ex_j &= \frac{(\eta_i - h_j)^2}{2(1 + b_j)(\eta_j - h_i)}; & Eu_j &= 0. \end{aligned}$$

⁵Clark and Riis (1995) solve the complete-information all-pay auction with a head start, and Fu (2006) examines the case of bias. Lemma 1 combines the techniques used in these papers.

4. If $h_i < h_j \leq \eta_j < \eta_i$, then the unique mixed-strategy equilibrium in the period-2 contest has expected efforts and payoffs given by

$$Ex_i = \frac{(\eta_j - h_j)(\eta_j + h_j - 2h_i)}{2(1 + b_i)(\eta_i - h_j)}; \quad Eu_i = 0;$$

$$Ex_j = \frac{(\eta_j - h_j)^2}{2(1 + b_j)(\eta_j - h_i)}; \quad Eu_j = \frac{\eta_i - \eta_j}{1 + b_j}.$$

5. If $\eta_j \leq h_j$, then $x_i = x_j = 0$; $Eu_i = 0$, and $Eu_j = V_j$.

The assumption that $(1 + b_i)V_i \geq (1 + b_j)V_j$ gives player i an edge over the rival in terms of the bias. In case 1 (5) player i (j) has a large head start that exceeds the hurdle set by the opponent; here, neither player exerts effort in the contest, resulting in the favored player winning and retaining the full prize value at the contest stage. Player i (j) has an advantage in case 2 (4) and gets a larger payoff than would occur in the all-pay auction without a head start or bias; there is fighting at the contest in these cases. Case 3 sees player j with an advantage, but not large enough to give an expected positive payoff. Since an advantage in our model needs to be bought by investment, player j would not be willing to invest to secure this outcome. Hence, player i (j) attempts to invest to achieve an outcome in the contest from case 1 or 2 (4 or 5).

3.2 First-period investments

By Lemma 1, one player will always expect a zero payoff at the contest stage; any investment by this player implies a negative overall expected payoff which is dominated by an investment of zero. This gives the following result:⁶

Lemma 2 (Sole investor). *There is no pure-strategy equilibrium in which both players simultaneously make non-zero investments.*

Using the definition of η_i in equation (1) together with Lemma 1, we can express player i 's expected net payoff as a function of his valuation, hurdle and investment

⁶See also Münster (2007a), Hirata (2014), Clark et al. (2025), and Clark et al. (2026).

levels:

$$W_i(V_i, \eta_i, h_i, b_i) = \begin{cases} V_i - c(h_i + b_i), & \text{if } \eta_i \leq h_i; \\ V_i - \frac{\eta_i - h_i}{1 + b_i} - c(h_i + b_i), & \text{if } h_i < \eta_i < h_i + (1 + b_i)V_i; \\ -c(h_i + b_i), & \text{if } \eta_i \geq h_i + (1 + b_i)V_i. \end{cases} \quad (2)$$

A useful way to see the investment incentives is to write the payoff as gross value minus the part not captured in the second-stage contest: $W_i = V_i - (V_i - Eu_i) - c(h_i + b_i)$, where V_i is the gross prize value and Eu_i is the continuation value characterized in Lemma 1. Thus, holding costs fixed, player i maximizes W_i by minimizing the *value lost* at the contest in stage 2, $V_i - Eu_i$. This loss equals 0 in the top line in (2), equals V_i in the bottom line in (2), and in the middle line (interior region), it takes the form

$$V_i - Eu_i = \frac{\eta_i - h_i}{1 + b_i}.$$

In the interior region, both instruments affect the value loss through the composite term $(\eta_i - h_i)/(1 + b_i)$ —increasing head start h_i lowers the numerator, while increasing bias b_i raises the denominator. These two instruments h_i and b_i are locally substitutable: increasing one instrument reduces the marginal return to the other. The proof of Lemma 3 formalizes this by showing that, away from a knife-edge, a small reallocation of investment from one instrument to the other, holding total investment fixed, strictly improves the objective, implying that optimal choices concentrate on a single instrument, either pure head start or pure bias. In the top line in (2), the hurdle is sufficiently low and it is optimal to set $h_i = \eta_i$ and choose no bias. In the bottom line in (2), no investment is a dominant strategy.

Lemma 3 (Single instrument). *Fix player i and parameters (V_i, η_i) , and consider the maximization of W_i in (2) over $(h_i, b_i) \in \mathbb{R}_+^2$.*

- (i) $(\eta_i, 0)$ weakly dominates any (h_i, b_i) such that $h_i \geq \eta_i$.
- (ii) $(0, 0)$ weakly dominates any (h_i, b_i) such that $h_i + (1 + b_i)V_i \leq \eta_i$.
- (iii) For (h_i, b_i) such that $0 \leq h_i < \eta_i < h_i + (1 + b_i)V_i$, holding total investment $y_i = h_i + b_i$ fixed, it is optimal to shift investment from bias to head start (from head start to bias) if $y_i > (<) \eta_i - 1$.

Consequently, player i 's optimal response to the hurdle η_i lies in $\{(0, 0), (h, 0), (0, b)\}$, for some strictly positive h, b .

Lemma 3 implies that, in computing best response functions, we can restrict attention to two one-dimensional problems, one for each instrument.⁷ Each of these single-instrument problems is strictly concave in its choice variable. We denote by h_i^* and b_i^* the single-instrument investment levels that maximize $W_i(V_i, \eta_i, h_i, 0)$ and $W_i(V_i, \eta_i, 0, b_i)$, respectively, with feasibility limits given by

$$h_i \in [L_i^H, U_i^H], b_i \in [L_i^B, U_i^B], \quad (3)$$

where

$$L_i^H := \max\{0, \eta_i - \min\{V_i, 1\}\}, \quad U_i^H := \eta_i; \quad (4)$$

$$L_i^B := \max\{0, \frac{\eta_i}{V_i} - 1\}, \quad U_i^B := \max\{0, \eta_i - 1\}. \quad (5)$$

These feasibility limits follow from the three cases on the right-hand side of (2) together with the optimal reallocation rule between the two instruments outlined in Lemma 3. We define the corresponding value functions of the single-instrument optimization problems as:

$$W_i^H(V_i, \eta_i) = W_i(V_i, \eta_i, h_i^*, 0); \quad (6)$$

$$W_i^B(V_i, \eta_i) = W_i(V_i, \eta_i, 0, b_i^*). \quad (7)$$

Having established the single-instrument property in Lemma 3, we next characterize the best-response candidates. Lemmas 4–5 solve the head-start-only and bias-only problems, project the interior maximizers onto the feasibility bounds, and rule out endpoints that are infeasible or cannot be optimal under the sole-investor restriction.

Lemma 4. *There are two possibilities for player i 's best response when the player's valuation is V_i and the hurdle the player faces is η_i .*

Case 1: $V_i < 1$.

- *If $W_i^H(V_i, \eta_i) > 0$, then player i 's best response is $(h_i, b_i) = (h_i^*, 0)$.*

⁷This facilitates comparison with single-instrument models, allowing us to draw out the specific strategic implication of the availability of two instruments. See also footnote 11.

- If $W_i^H(V_i, \eta_i) \leq 0$, then player i 's best response is $(h_i, b_i) = (0, 0)$.

Case 2: $V_i \geq 1$.

- If $W_i^H(V_i, \eta_i) > \max\{0, W_i^B(V_i, \eta_i)\}$, then player i 's best response is $(h_i, b_i) = (h_i^*, 0)$.
- If $W_i^B(V_i, \eta_i) > \max\{0, W_i^H(V_i, \eta_i)\}$, then player i 's best response is $(h_i, b_i) = (0, b_i^*)$.
- If $W_i^H(V_i, \eta_i) = W_i^B(V_i, \eta_i) > 0$, then player i 's best response is either $(h_i, b_i) = (h_i^*, 0)$ or $(h_i, b_i) = (0, b_i^*)$.
- If $\max\{W_i^H(V_i, \eta_i), W_i^B(V_i, \eta_i)\} \leq 0$, then player i 's best response is $(h_i, b_i) = (0, 0)$.

Lemma 4 reflects the piecewise structure of W_i in (2). In particular, an interior optimum for a given instrument is evaluated using the middle line of (2), whereas a boundary choice (such as $h_i = U_i^H = \eta_i$) is evaluated using the top line. To characterize the maximizers of the one-dimensional problems, it is useful to define the unconstrained interior candidates. Let h^\diamond denote the interior optimizer of the head-start-only problem, and let $b^\diamond(\eta_i)$ denote the interior optimizer of the bias-only problem:

$$h^\diamond := \{h | c'(h) = 1\} > 0; \text{ and} \quad (8)$$

$$b^\diamond(\eta_i) := \{b | (1+b)^2 c'(b) = \eta_i\} > 0, \quad i = 1, 2. \quad (9)$$

Since $(1+b)^2 c'(b)$ is strictly increasing in b , the solution $b^\diamond(\eta_i)$ is uniquely defined and strictly increasing in η_i .

It is worth noting that the best-response candidates derived in Lemmas 3 and 4 do not use the sole-investor property. This property, as shown in Lemma 2, holds in any equilibrium. Following Lemma 2, we now focus on equilibria in which only one player invests, which further simplifies the set of best-response candidates. If player i is the sole investor, then $h_j = b_j = 0$, and hence $\eta_i = V_j$ for $j \neq i$. The following lemma proves that any action taken in equilibrium that is an endpoint of the feasible sets in (4) and (5) will be independent of a player's own valuation. Furthermore, since the interior maximizers are also independent of a player's own valuation, it follows that

any action chosen by player i in response to the hurdle $\eta_i = V_j$ will be independent of V_i .

Lemma 5. *Fix a valuation profile (V_i, V_j) . Suppose player i is the sole investor, so that $h_j = b_j = 0$, and hence $\eta_i = V_j$. Then the single-instrument investment levels h_i^* and b_i^* that maximize $W_i(V_i, V_j, h_i, 0)$ and $W_i(V_i, V_j, 0, b_i)$ are given, respectively, by*

$$h_i^* = \min\left\{V_j, \max\{V_j - 1, h^\diamond\}\right\} = \begin{cases} V_j, & \text{if } V_j < h^\diamond, \\ h^\diamond, & \text{if } h^\diamond \leq V_j \leq 1 + h^\diamond, \\ V_j - 1, & \text{if } V_j > 1 + h^\diamond, \end{cases} \quad (10)$$

$$b_i^* = \min\left\{\max\{0, V_j - 1\}, b^\diamond(V_j)\right\} = \begin{cases} 0, & \text{if } V_j < 1, \\ V_j - 1 & \text{if } 1 \leq V_j \leq \hat{b}, \\ b^\diamond(V_j), & \text{if } V_j > \hat{b}, \end{cases} \quad (11)$$

where h^\diamond and $b^\diamond(\cdot)$ are defined in (8) and (9); \hat{b} uniquely solves

$$bc'(b - 1) = 1; \quad (12)$$

and $1 < \hat{b} < 1 + h^\diamond$.

Both maximizers are continuous and weakly increasing in V_j . Moreover, the difference between the value functions of the two single-instrument problems, $W_i^H(V_i, V_j) - W_i^B(V_i, V_j)$, can be written as a function of V_j alone,

$$\Delta(V_j) := W_i^H(V_i, V_j) - W_i^B(V_i, V_j) = h_i^* - V_j - c(h_i^*) + \frac{V_j}{1 + b_i^*} + c(b_i^*), \quad (13)$$

and is therefore independent of V_i .

Lemma 5 reduces the equilibrium analysis to comparing the optimized head-start and bias payoffs as functions of the opponent's valuation alone. The next lemmas exploit this reduction. Lemma 6 outlines an important indifference result that informs a player's optimal choice between the instruments, while Lemma 7 rules out non-viable candidate endpoints.

Lemma 6. *Suppose player i is the sole investor and fix V_j . Player i is indifferent between using the two instruments if $\Delta(V_j) = 0$. Moreover, there exists a unique*

$\tilde{V} \in (\hat{b}, 1 + h^\diamond)$ such that $\Delta(\tilde{V}) = 0$, and $\Delta(V_j) > 0$ for $V_j \in (0, \tilde{V})$ and $\Delta(V_j) < 0$ for $V_j > \tilde{V}$.

Lemma 6 provides a powerful result: in any equilibrium in which only player i invests, his optimal instrument choice follows a cut-off rule—he uses a head start when the valuation of the rival, V_j , is below a threshold and bias when V_j is above it. We can also rule out the lower endpoint $V_j - 1$ as an equilibrium investment, whether chosen as a head start or as a bias, since this gives a negative profit.

Lemma 7. *Consider $V_j > 1$. Neither $(h_i = V_j - 1, b_i = 0)$ nor $(h_i = 0, b_i = V_j - 1)$ can be part of a pure-strategy equilibrium in which player i is the sole investor.*

In equilibrium, player i must also set the chosen instrument so that the rival does not have a profitable deviation from zero investment to some positive investment (head start or bias). Suppose player i invests in such a way that player j faces a hurdle η_j . By Lemma 3, we can restrict attention to deviations by player j using a single instrument. To this end, we define the following deviation payoffs for player j :

$$D^H(\eta_j, V_j) := \max_{h \in [L_j^H, U_j^H]} \{V_j - \eta_j + h - c(h)\}, \quad (14)$$

$$D^B(\eta_j, V_j) := \max_{b \in [L_j^B, U_j^B]} \{V_j - \frac{\eta_j}{1+b} - c(b)\}, \quad (15)$$

where the feasibility limits, which are functions of η_j and V_j , are those given in (3) (up to a change in index). Lemma 8 outlines conditions that must be fulfilled to prevent player i 's rival from deviating from a no-investment strategy.

Lemma 8. *Let player i invest and let η_j be the hurdle for player j . The head-start (bias) no-deviation threshold, i.e., the minimum η_j that prevents j from profitably*

deviating from a no-investment strategy by investing in head start (bias), is given by

$$\begin{aligned} \eta^H(V_j) &:= \inf\{\eta \geq 0 : D^H(\eta, V_j) \leq 0\} \\ &= \begin{cases} c^{-1}(V_j), & \text{if } 0 < V_j < c(h^\diamond), \\ V_j + h^\diamond - c(h^\diamond), & \text{if } c(h^\diamond) \leq V_j \leq 1 + c(h^\diamond), \\ 1 + c^{-1}(V_j - 1), & \text{if } V_j > 1 + c(h^\diamond), \end{cases} \end{aligned} \quad (16)$$

$$\begin{aligned} \eta^B(V_j) &:= \inf\{\eta \geq 0 : D^B(\eta, V_j) \leq 0\} \\ &= \begin{cases} 0, & \text{if } 0 < V_j < 1 \\ 1 + c^{-1}(V_j - 1), & \text{if } 1 \leq V_j \leq 1 + c(\hat{b} - 1), \\ (1 + b^\dagger(V_j))(V_j - c(b^\dagger(V_j))), & \text{if } V_j > 1 + c(\hat{b} - 1), \end{cases} \end{aligned} \quad (17)$$

where $b^\dagger(V_j) > 0$ uniquely solves

$$(1 + b) c'(b) = V_j - c(b). \quad (18)$$

The two thresholds $\eta_j^B(V_j)$ and $\eta_j^H(V_j)$ are non-decreasing in V_j ; $\eta_j^H(V_j)$ is continuous for $V_j \geq 0$, and $\eta_j^B(V_j)$ is continuous for $V_j \geq 1$.

Player j has no profitable deviation from $(h_j, b_j) = (0, 0)$ if and only if

$$\eta_j \geq \max\{\eta^B(V_j), \eta^H(V_j)\}.$$

The thresholds $\eta^H(V_j)$ and $\eta^B(V_j)$ are the minimum values of the hurdle η_j that render j 's most profitable feasible head-start deviation and bias deviation, respectively, unprofitable.⁸ These thresholds are piecewise because the optimal deviation is not always the interior maximizer of player j 's deviation payoff. Economically, the two threshold functions summarize the rival's outside options: they measure how much of an induced advantage is required to make deviation through each investment margin unprofitable.

Consider first the head-start threshold. If player j deviates from no investment by choosing a head start, and the interior optimum h^\diamond is feasible, then the most profitable deviation payoff is $V_j + h^\diamond - c(h^\diamond) - \eta_j$; setting this to zero gives the

⁸The threshold that prevents player j from deviating from the no-investment strategy depends only on V_j . For this reason, these thresholds do not carry a player-specific subscript.

threshold $\eta^H = V_j + h^\diamond - c(h^\diamond)$.

For the bias threshold, the construction is analogous but reflects the multiplicative structure of the instrument. If player j deviates by choosing a bias, $b^\dagger(V_j)$ identifies the smallest bias level at which the interior optimal bias deviation yields a non-negative payoff. When this interior optimum is feasible, setting the maximized deviation payoff to zero gives the deterrence threshold $\eta^B = (1 + b^\dagger(V_j))(V_j - c(b^\dagger(V_j)))$.

When these interior choices are not feasible given the relevant constraints, player j is forced to deviate at a constrained boundary point, which is less profitable than the unconstrained optimum. This additional constraint works in favor of the investing player i : the zero-payoff condition, now imposed at the boundary maximizer, is satisfied at a lower hurdle, so deviations from zero investment are less profitable for player j .

Finally, an investment profile by player i gives no profitable deviations by player j if and only if the induced hurdle satisfies $\eta_j \geq \max\{\eta^H(V_j), \eta^B(V_j)\}$, simultaneously blocking both a head-start and a bias deviation.

The structure of the two thresholds has a natural economic interpretation that varies with the rival's valuation.⁹ When V_j is modest, the head-start threshold $\eta^H(V_j)$ binds: the rival's most attractive deviation is to secure an additive advantage, since the prize is not large enough to justify investing in effort productivity. As V_j rises, the bias threshold $\eta^B(V_j)$ eventually becomes binding, because the multiplicative return to bias scales with the value of winning and the rival finds it increasingly attractive to invest in effort productivity rather than position. The crossover point at which $\eta^B(V_j)$ overtakes $\eta^H(V_j)$ therefore marks a qualitative shift in the investor's deterrence problem: below it, the investor need only neutralize an additive threat; above it, a multiplicative threat becomes the binding constraint.

This distinction has concrete implications in applications. In an R&D race, for example, an investor who builds laboratory infrastructure (a head start) faces a risk that the rival will respond by copying this head start investment when the rival values the patent modestly. But when the prize is large, the rival's incentive to invest in researcher training (bias) becomes dominant, and the investor must generate a sufficiently large hurdle to make even that more productive deviation unprofitable. In sales contests, a firm that invests in customer pipelines faces an analogous escalation: against a rival with a small territory, pipeline investment deters head-start copying;

⁹See the example in Section 4.

against a rival with a large territory, it must also deter investment in sales training. The investor's problem thus becomes harder as the rival's valuation rises not only because the rival fights harder at the contest stage, but also because the nature of the rival's most attractive pre-contest deviation changes.

Lemmas 1–8 establish two building blocks for characterizing the pure-strategy subgame-perfect Nash equilibria (SPNE): the continuation values that govern each player's incentive to invest, and the thresholds that rule out profitable deviations by the non-investor. Proposition 1 collects the results and states the pure-strategy SPNE of this game.

Proposition 1 (Pure-strategy SPNE). *Every pure-strategy SPNE has exactly one investor in period 1. Fix $i \neq j$, and consider a candidate equilibrium in which player i is the sole investor, so that $h_j = 0$, $b_j = 0$, and player i faces the hurdle $\eta_i = V_j$.*

(a) **Existence.** *There exists a pure-strategy equilibrium of this form in which only player i invests if and only if*

$$\max\{W_i^H(V_i, V_j), W_i^B(V_i, V_j)\} > 0,$$

and the induced hurdle for player j , η_j , satisfies

$$\eta_j \geq \max\{\eta^H(V_j), \eta^B(V_j)\}.$$

(b) **Optimal instrument.** *Player i optimally uses a single instrument: there exists a cutoff \tilde{V} , given by $\Delta(\tilde{V}) = W_i^H(V_i, \tilde{V}) - W_i^B(V_i, \tilde{V}) = 0$, such that head start is optimal for $V_j < \tilde{V}$, bias is optimal for $V_j > \tilde{V}$, and player i is indifferent at $V_j = \tilde{V}$. The equilibrium investment profile satisfies*

$$(h_i, b_i) = \begin{cases} (\min\{V_j, h^\circ\}, 0), & \text{if } V_j < \tilde{V}; \\ (0, b^\circ(V_j)), & \text{if } V_j > \tilde{V}; \\ \text{either of the above (indifference),} & \text{if } V_j = \tilde{V}; \end{cases} \quad (19)$$

where h° and $b^\circ(\cdot)$ are given in (8) and (9).

(c) **Second-period play.** *Given period-1 investments, period-2 play follows Lemma 1. If $h_i = V_j$ in equilibrium, then both players exert zero effort in the contest. Otherwise, the second period contest exhibits positive effort as characterized in Lemma 1.*

Proposition 1 gives the salient features of an SPNE. Exactly one player will invest; this can either be the player with the larger or the smaller valuation of winning the contest.¹⁰ An equilibrium in which neither player invests cannot arise: since $c'(0) = 0$, any player facing the no-investment contest can profitably deviate by choosing a small positive head start, the marginal gain from which exceeds the marginal cost at zero. Hence, every pure-strategy SPNE involves exactly one investor.

Furthermore, the investing player uses a single instrument. Economically, the result reflects that pre-contest investment is used to influence the rival's continuation incentives rather than to accumulate advantage along every available margin; with a common convex investment technology, concentrating investment on the most effective single margin is optimal.¹¹

The choice of instrument by the investing player depends only on the valuation of the rival. If the rival has a low valuation, then this player is already quite weak at the contest stage. The investor uses a head-start investment in this case, following equation (19), so that the investment level will be $\min\{V_j, h^\diamond\}$. For very low opponent values, $h_i^* = V_j$ and there will be no fighting in the contest; as the opponent's value rises, $h_i^* = h^\diamond$, which is a constant, and the contest will be fought. As V_j increases beyond the cutoff \tilde{V} , the investor switches to bias, choosing $b_i^* = b^\diamond(V_j)$; effort is also in this case exerted in the contest. The intuition is that additive advantage is the cheaper way to neutralize weak rivals, while multiplicative advantage becomes more effective once the rival values the prize enough that simply moving the starting line is no longer the best way to make deviations unattractive.

Although the investor's problem always has a profitable single-instrument solution against a non-investing rival, a pure-strategy SPNE need not exist.¹² Non-existence

¹⁰Section 4 demonstrates that parameter regions can arise which support the coexistence of an equilibrium in which the stronger player invests and one in which the weaker player invests. Standard refinements do not discriminate between them: as shown in Clark et al. (2026) for the head-start instrument, both equilibrium types satisfy weak perfection.

¹¹The single-instrument property relies on the shared cost structure $c(h_i + b_i)$, which makes head start and bias substitutable in cost space. Under an additively separable cost $c_h(h_i) + c_b(b_i)$, the two first-order conditions become decoupled and an interior mixture of both instruments could in principle be optimal, since reallocating investment between instruments no longer holds total cost fixed. The cutoff rule characterizing instrument choice in Proposition 1(b) would then cease to apply, and the analysis of equilibrium existence would require tracking a two-dimensional investment choice. We conjecture that the core qualitative finding—that the investor concentrates on the instrument better suited to deterring the rival's preferred deviation—would survive under separable costs, but the boundary between the head-start and bias regions would shift and the single-instrument property itself may not hold exactly. We leave this extension for future work.

¹²A mixed strategy equilibrium may exist. See the discussion in Clark et al. (2026) for the head

arises when the investor's preferred instrument against no investment leaves the rival with a profitable deviation through at least one investment margin. The quadratic-cost case in Section 4 illustrates this mechanism and shows how equilibrium can move from head start to non-existence, then to bias, and finally back to non-existence as the rival's valuation increases.

4 Illustration: quadratic cost function

To illustrate the results from the general model, we suppose $c(h + b) = \frac{k}{2}(h + b)^2$, $k > 0$. Recall from the general analysis in Section 3 that h° , $b^\circ(\cdot)$, $b^\dagger(\cdot)$, and the deviation thresholds $\eta^H(\cdot)$ and $\eta^B(\cdot)$ are defined in equations (8), (9), (12), (16), and (17). With the quadratic cost function, they take the following closed forms:

$$h^\circ = \frac{1}{k}; \quad \hat{b} = \frac{1 + \sqrt{1 + 4/k}}{2}; \quad b^\dagger(V) = \frac{-1 + \sqrt{1 + 6V/k}}{3}. \quad (20)$$

$$\eta^H(V) = \begin{cases} \sqrt{\frac{2V}{k}}, & \text{if } V < \frac{1}{2k}; \\ V + \frac{1}{2k}, & \text{if } \frac{1}{2k} \leq V \leq 1 + \frac{1}{2k}; \\ 1 + \sqrt{\frac{2(V-1)}{k}}, & \text{if } V > 1 + \frac{1}{2k}. \end{cases} \quad (21)$$

$$\eta^B(V) = \begin{cases} 0 & \text{if } V < 1; \\ 1 + \sqrt{\frac{2(V-1)}{k}}, & \text{if } 1 < V \leq 1 + \frac{k(\hat{b}-1)^2}{2}; \\ kb^\dagger(V)(1 + b^\dagger(V))^2, & \text{if } V > 1 + \frac{k(\hat{b}-1)^2}{2}. \end{cases} \quad (22)$$

The function $b^\circ(V) > 0$ and the cutoff $\tilde{V}(k) \in (\hat{b}, 1 + 1/k)$ satisfy the equations

$$\begin{aligned} b^\circ(V) : \quad & kb(1 + b)^2 = V, \\ \tilde{V}(k) : \quad & \min\left\{\frac{1}{k}, V\right\} - V - c\left(\min\left\{\frac{1}{k}, V\right\}\right) + \frac{V}{1 + b^\circ(V)} - b^\circ(V) = 0, \end{aligned}$$

start instrument.

respectively. Furthermore, to simplify the characterization of the SPNE, we define the auxiliary functions

$$\begin{aligned}\phi_1(V, k) &:= \sqrt{\frac{2V}{k}} - V, & \phi_2(V, k) &:= \max\{\eta^H(V), \eta^B(V)\} - \min\{1/k, V\}, \\ \phi_3(V, k) &:= \frac{\max\{\eta^H(V), \eta^B(V)\}}{1 + b^\circ(V)}, & \phi_4(V, k) &:= \max\left\{\frac{1}{2k}, V - \frac{1}{2k}\right\}.\end{aligned}$$

We first present a reinforcement SPNE in which the high-valuation player 1 is the sole investor; investment amplifies the strong player's initial advantage in the contest. Then we give a preemptive equilibrium, in which the low-valuation player 2 invests to alleviate the initial disadvantage by securing a boost before the contest stage.

4.1 Reinforcement investment

Corollary 1 outlines some special features of the equilibrium with reinforcement investment.

Corollary 1. *Fix $V_1 > V_2 > 0$, and consider a candidate reinforcement equilibrium in pure strategies in which player 1 is the sole investor.*

A. *Suppose $0 < V_2 < 1$. The reinforcement equilibrium, if it exists, has player 1 investing in a head start.*

(i) *For $V_2 \in (\frac{1}{2k}, 1)$ (which is non-empty only if $k \geq \frac{1}{2}$), the reinforcement equilibrium exists with $(h_1, b_1) = (\min\{\frac{1}{k}, V_2\}, 0)$.*

(ii) *For $V_2 < \min\{\frac{1}{2k}, 1\}$, the reinforcement equilibrium exists with $(h_1, b_1) = (V_2, 0)$ if and only if $V_1 \geq \phi_1(V_2, k)$. The existence condition imposes a binding restriction and it holds only for sufficiently small V_2 .*

(iii) *The non-existence region in (V_1, V_2) -space given by $I := \{(V_1, V_2) \mid 0 < V_2 < \min\{V_1, 1, \frac{1}{2k}\}, V_1 < \phi_1(V_2, k)\}$ is non-empty for all $k > 0$ and shrinks as k increases.*

B. *Suppose $V_2 \geq 1$. The reinforcement equilibrium, if it exists, has player 1 investing in either a head start or a bias.*

(i) **Head-start equilibrium.** *For $1 \leq V_2 \leq \tilde{V}(k)$, the reinforcement equilibrium exists with $(h_1, b_1) = (\min\{\frac{1}{k}, V_2\}, 0)$ if and only if $V_1 \geq \phi_2(V_2, k)$.*

(ii) **Bias equilibrium.** For $V_2 > \tilde{V}(k)$, the reinforcement equilibrium exists with $(h_1, b_1) = (0, b^\circ(V_2))$ if and only if $V_1 \geq \phi_3(V_2, k)$.

(iii) The non-existence region in (V_1, V_2) -space given by $II := \{(V_1, V_2) \mid 1 \leq V_2 < \tilde{V}_2(k), V_2 < V_1, V_1 < \phi_2(V_2, k)\}$ can be an empty set.

(iv) The non-existence region in (V_1, V_2) -space given by $III := \{(V_1, V_2) \mid V_2 > \tilde{V}_2(k), V_2 < V_1, V_1 < \phi_3(V_2, k)\}$ is non-empty for a sufficiently large V_2 .

Economically, Corollary 1 shows that the strong player's problem is not simply to invest profitably, but to invest in a way that survives the rival's no-deviation constraints. Against a weak rival, this is most cheaply done through a head start. When the rival's valuation increases, the same additive advantage is no longer enough, and the strong player eventually prefers bias because it more effectively reduces the rival's gain from challenging. The non-existence regions arise because player 1's preferred investment against a passive rival need not be strong enough to leave player 2 with no profitable deviation. Thus reinforcement fails not because investment is unprofitable per se, but because the investment that is privately optimal for player 1 does not always satisfy the rival's two no-deviation constraints.

Corollary 1A considers $0 < V_2 < 1$, where player 1 uses a head start, and gives player 2 an effective hurdle of $\eta_2 = V_1 + \min\{V_2, 1/k\}$. Lemma 8 implies that player 2's most profitable deviation is through a head start, yielding a no-deviation threshold $\eta^H(V_2) = \min\{\sqrt{2V_2/k}, V_2 + 1/(2k)\}$. When $1/(2k) \leq V_2 < 1$, the condition for unprofitable deviations $\eta_2 \geq \eta^H(V_2)$ is automatically satisfied for all $V_1 > V_2$. When $0 < V_2 < 1/(2k)$, the no-deviation condition reduces to $V_1 > \phi_1(V_2, k)$, so the strong player's valuation must be sufficiently large to prevent profitable deviations. As k increases (investment becomes more costly), the non-existence region near the origin shrinks, since costlier deviations are more likely to be unprofitable. Region I in both panels of Figure 1 illustrates this set of parameter values, where player 2 has a profitable deviation to $h_2 = V_1 + V_2$.

Part B(i) considers $V_2 \in [1, \tilde{V}(k)]$, where player 1 continues to reinforce his advantage through a head start. However, player 2's most profitable deviation may now arise through either instrument. Accordingly, the no-deviation condition reduces to $\eta_2 = V_1 + \min\{1/k, V_2\} \geq \max\{\eta^H(V_2), \eta^B(V_2)\}$, i.e. $V_2 \geq \phi_2(V_2, k)$. Both the induced hurdle η_2 and the no-deviation thresholds are increasing in V_2 , so the measure of the non-existence region varies non-monotonically with V_2 and may even be empty for

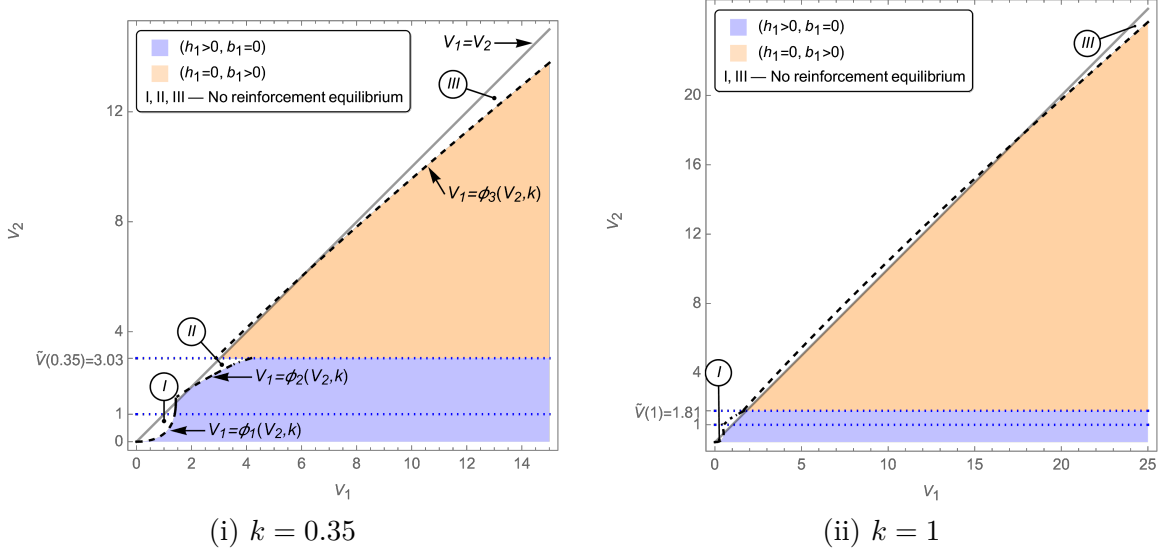


Figure 1: Equilibrium with reinforcement investment.

some parameter values, as illustrated in panel (ii) of Figure 1.

For $V_2 > \tilde{V}(k)$, player 1 uses bias as the reinforcing instrument, with the candidate reinforcement equilibrium $(h_1, b_1) = (0, b^\circ(V_2))$. The no-deviation condition is $V_1(1 + b^\circ(V_2)) \geq \max\{\eta^H(V_2), \eta^B(V_2)\}$, i.e. $V_2 \geq \phi_3(V_2, k)$. For large enough V_2 , reinforcement with bias is feasible only when V_1 is sufficiently high relative to V_2 ; otherwise, even an optimally chosen bias by player 1 cannot prevent player 2 from deviating.

Figure 1 illustrates the equilibrium regions and the corresponding non-existence regions I, II, and III, for two values of the cost parameter $k \in \{0.35, 1\}$. The dashed curves represent the thresholds $\phi_1(V_2, k)$, $\phi_2(V_2, k)$, and $\phi_3(V_2, k)$. In the blue region, player 1 uses a head start, setting $h_1 = V_2$ when $V_2 < 1/k$ and $h_1 = 1/k$ when $1/k \leq V_2 \leq \tilde{V}(k)$; in the orange region, player 1 uses bias, setting $b_1 = b^\circ(V_2)$. The two regions are separated by the horizontal cutoff $\tilde{V}(k)$.

Non-existence in region I occurs when both valuations are small and investment costs are low so player 1's head start cannot prevent player 2 from investing.¹³ Region III arises for large V_2 , where even player 1's optimal bias is insufficient to prevent deviation. Region II, which may be empty, arises for intermediate V_2 when player 1's head start fails to render player 2's most profitable deviation unprofitable.

¹³This region is always present but is small in panel (ii) as the axes are plotted on a wider scale.

4.2 Preemptive investment

Corollary 2 considers equilibrium with preemptive investment. Intuitively, the weak player can sustain equilibrium only by choosing an investment that changes the strong rival's incentives enough to make challenge unattractive. This is easiest when valuations are close, and it becomes impossible once the valuation gap is too large. Against a not-too-strong rival, preemption takes the form of a head start, since an additive advantage is the cheapest way to overturn the initial disadvantage. As V_1 rises, player 2 eventually switches to bias, because improving the productivity of effort becomes a more effective way to withstand challenge by a stronger opponent.

Corollary 2. *Fix $V_1 > V_2 > 0$ and consider a candidate preemptive equilibrium in pure strategies in which player 2 is the sole investor.*

A. *Suppose $0 < V_1 < 1$. The preemptive equilibrium, if it exists, has player 2 investing in a head start.*

(i) *For all $V_1 \in (\frac{1}{2k}, 1)$ (which is non-empty only if $k \geq \frac{1}{2}$), the preemptive equilibrium exists with $(h_2, b_2) = (\min\{1/k, V_1\}, 0)$ if and only if $V_2 \geq \phi_4(V_1, k)$.*

(ii) *For all $V_1 < \min\{\frac{1}{2k}, 1\}$, no preemptive equilibrium exists.*

(iii) *The non-existence region in (V_1, V_2) -space given by $IV := \{(V_1, V_2) \mid 0 < V_2 < V_1 < 1, V_1 > \frac{1}{2k}, V_2 < \phi_4(V_1, k)\} \cup \{(V_1, V_2) \mid 0 < V_2 < V_1 \leq \frac{1}{2k}\}$ is always non-empty for any given $k > 0$.*

B. *Suppose $V_1 \geq 1$. The preemptive equilibrium, if it exists, has player 2 investing in either a head start or a bias.*

(i) **Head-start equilibrium.** *For $1 \leq V_1 \leq \tilde{V}(k)$, the preemptive equilibrium exists with $(h_2, b_2) = (\min\{1/k, V_1\}, 0)$ if and only if $V_2 \geq \phi_2(V_1, k)$.*

(ii) **Bias equilibrium.** *For $V_1 > \tilde{V}(k)$, the preemptive equilibrium exists with $(h_2, b_2) = (0, b^\circ(V_1))$ if and only if $V_2 \geq \phi_3(V_1, k)$.*

(iii) *The non-existence region in (V_1, V_2) -space given by $V := \{(V_1, V_2) \mid 1 \leq V_1 < \tilde{V}(k), V_2 < V_1, V_2 < \phi_2(V_1, k)\}$ is always non-empty for any given $k > 0$.*

(iv) *The non-existence region in (V_1, V_2) -space given by $VI := \{(V_1, V_2) \mid V_1 > \tilde{V}(k), V_2 < V_1, V_2 < \phi_3(V_1, k)\}$ is always non-empty for any given $k > 0$.*

Corollary 2A considers $0 < V_1 < 1$, so a head start is the only feasible instrument for both players. When V_1 is at least $1/(2k)$, player 2's valuation must be sufficiently large in absolute terms and sufficiently close to player 1's valuation to preclude profitable deviations by player 1. By contrast, when player 1's valuation falls below $1/(2k)$, ensuring that player 1 does not have a profitable deviation from $(h_1, b_1) = (0, 0)$ requires $V_2 > \phi_4(V_1, k)$ (following a similar mechanism as shown in establishing the existence of the reinforcement equilibrium) that is increasing in player 1's valuation. Since this threshold is increasing, its minimum is attained when player 1's valuation equals player 2's, in which case the threshold strictly exceeds player 2's own valuation. Player 1 thus has a profitable deviation from no investment whenever both players' valuations fall below $1/(2k)$.

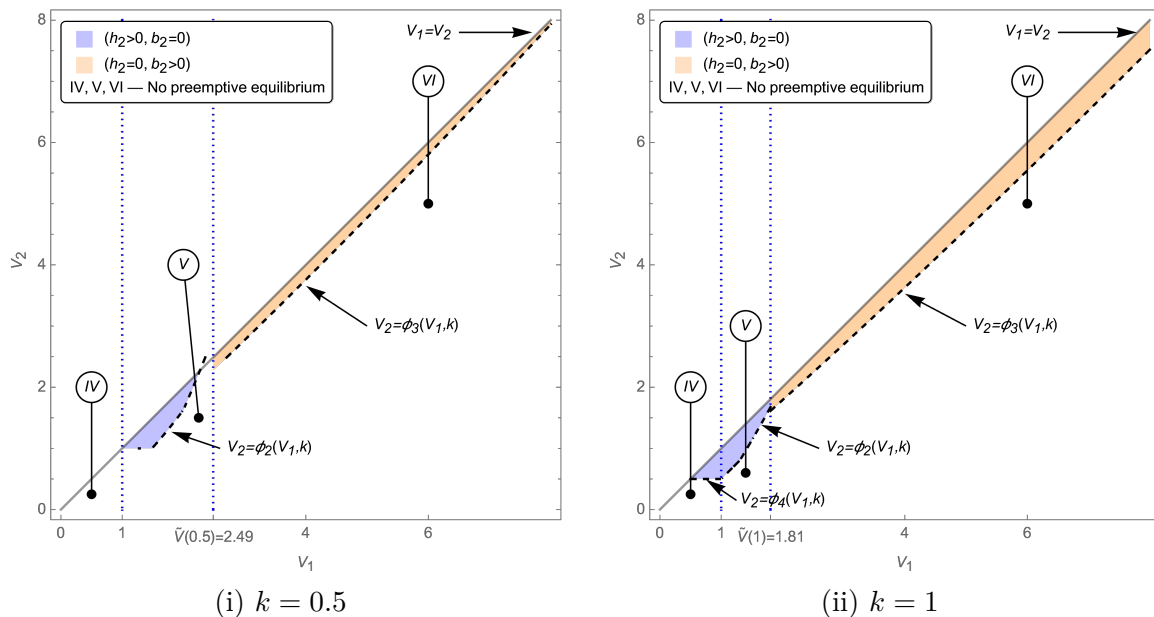


Figure 2: Equilibrium with preemptive investment.

Part B(i) considers $V_1 \in [1, \tilde{V}(k)]$. For V_1 below this cutoff, player 2 continues to invest in a head start, making the effective hurdle faced by player 1 $\eta_1 = V_2 + \min\{V_1, 1/k\}$. The stronger player may profitably deviate using either instrument, making the no-deviation condition: $V_2 + \min\{1/k, V_1\} \geq \max\{\eta^H(V_1), \eta^B(V_1)\}$. When the rival's valuation exceeds the cutoff, we are in part B(ii) of Corollary 2, and bias is used player 2: the candidate equilibrium is $(h_2, b_2) = (0, b^\circ(V_1))$, and the no-deviation constraint for player 1 reduces to $V_2(1 + b^\circ(V_1)) \geq \max\{\eta^H(V_1), \eta^B(V_1)\}$. As V_1 grows large relative to V_2 , sustaining such an equilibrium becomes increasingly difficult. In

particular, preemption can be supported only when the two valuations are sufficiently close; otherwise, player 1 retains a profitable deviation.

Figure 2 depicts the equilibrium configuration together with the non-existence regions IV, V, and VI for two values of the cost parameter $k \in \{0.5, 1\}$. The dashed curves correspond to the boundaries $\phi_4(V_1, k)$, $\phi_2(V_1, k)$, and $\phi_3(V_1, k)$. The blue region represents valuations for which player 2 optimally employs a head start. In this region, the optimal investment equals $h_2 = V_1$ when $V_1 < 1/k$, and $h_2 = 1/k$ once the cap becomes binding. The orange region corresponds to equilibria in which bias is used: $b_2 = b^\diamond(V_1)$. The two regimes are separated by the vertical threshold $\tilde{V}(k)$.

Region IV appears near the origin, where either player 2's valuation is too small or the valuation gap is too large for a head start to prevent deviations by the stronger rival. Regions V and VI arise when V_1 becomes sufficiently large relative to V_2 . Panel (i) of Figure 2 shows that the equilibrium regions are disconnected. Between the blue and orange areas, player 2 would optimally choose the capped head-start investment $h_2 = 1/k$, but this level is insufficient to prevent player 1 from deviating from no investment to a positive level of bias. Although the head start raises the effective hurdle faced by player 1, the attractiveness of bias increases with V_1 while its cost remains low when k is small. Consequently, player 1 can profitably deviate from no investment, and no pure-strategy SPNE with player 2 investing exists in this intermediate region. Once V_1 exceeds $\tilde{V}(k)$, however, bias becomes sufficiently effective and the preemptive equilibrium is restored.

5 Instrument availability and equilibrium existence

Suppose now that players have access to only a single instrument—either head start or bias, but not both. Clark et al. (2026) show that, in the head-start-only case, any subgame-perfect equilibrium with preemptive investment by the weak player (player 2) is accompanied by an equilibrium in which the strong player (player 1) invests instead (reinforcement). Clark et al. (2025) obtain the same existence property in a bias-only model with linear investment costs. Proposition A.1 in the Appendix extends this single-instrument existence result to strictly convex investment costs. The monotonicity property arises with availability of a single instrument since the induced hurdle varies continuously and monotonically in the investor's valuation.

Below, we show that there is a jump in the induced hurdle when a player transitions between instruments, and this can lead to the existence of valuation pairs (V_1, V_2) that support an equilibrium in which only the weaker player invests without admitting an equilibrium in which the stronger player is the investor. We establish this first by example, and then provide a sufficient condition that is directly related to properties of the cost function.

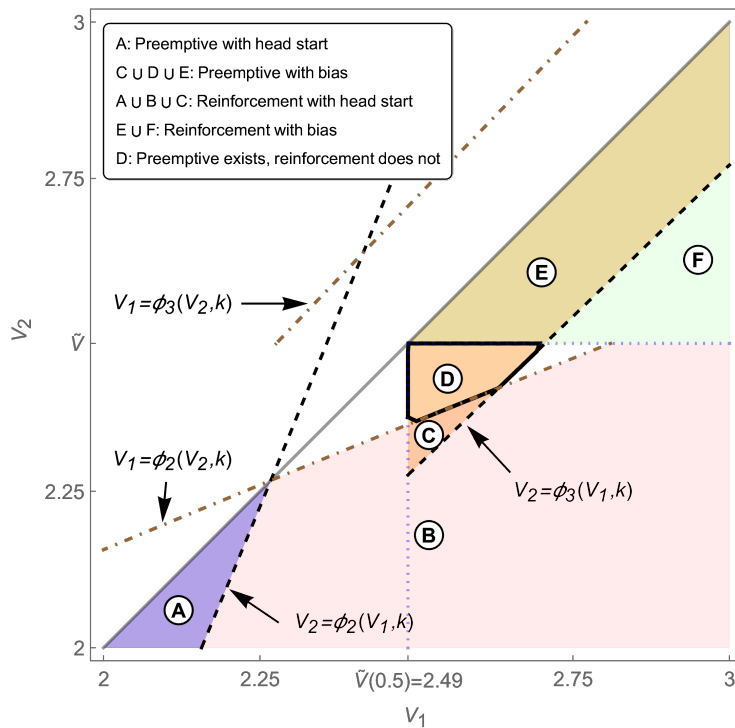


Figure 3: Existence of preemptive but not reinforcement equilibrium, $k = 0.5$

In the example, we retain quadratic investment costs, and set $k = 0.5$, so that $\hat{b} = h^\diamond = 2$, and $\tilde{V}(0.5) \approx 2.49$. Figure 3 illustrates the equilibrium regions for $V_i \in (\hat{b}, 1 + h^\diamond) = (2, 3)$. Preemptive equilibrium exists in regions A and $C \cup D \cup E$, with player 2 using head start below \tilde{V} and bias above \tilde{V} . Reinforcement equilibrium exists in regions $A \cup B \cup C$ and $E \cup F$, with player 1 again using head start below \tilde{V} and bias above \tilde{V} . The key region is D : here, player 2 can profitably invest in bias, but player 1 has no profitable reinforcement equilibrium. Thus, once both instruments are available, preemption no longer implies reinforcement.

The comparison underscores that a two-instrument game cannot simply be treated as a single-instrument contest with an endogenous choice of instrument. Even though

we have shown that the investor uses only one instrument in equilibrium, any candidate investment must ensure that the opponent has no profitable deviation in both dimensions—head start and bias. The economic difference is that, with one instrument, equilibrium existence is ordered by strength, whereas with two instruments it depends on how well a chosen form of investment targets the rival’s most attractive deviation. The following proposition gives a sufficient condition for the existence of valuations that support a preemptive equilibrium, but not reinforcement.

Proposition 2. *Define $\bar{\eta} := \max\{\eta^H(\tilde{V}), \eta^B(\tilde{V})\}$. A sufficient condition for the existence of a valuation profile that support a preemptive equilibrium but no reinforcement equilibrium is*

$$\tilde{V} + \min\{\tilde{V}, h^\circ\} < \bar{\eta} < \tilde{V}(1 + b^\circ(\tilde{V})). \quad (23)$$

Proposition 2 identifies the discontinuous jump in the induced hurdle at the instrument-switching point \tilde{V} as the proximate cause of monotonicity failure.

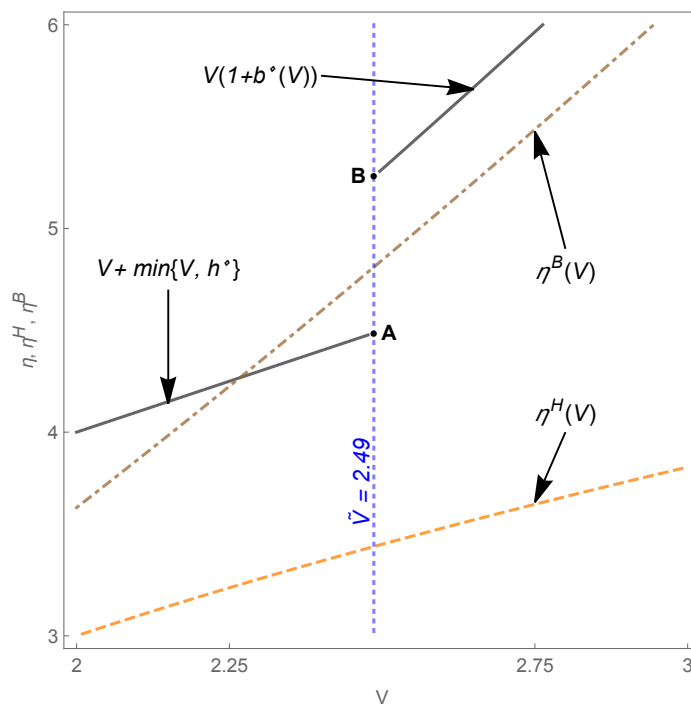


Figure 4: Sufficiency condition

When the rival’s valuation crosses \tilde{V} , the investor’s optimal instrument switches from head start to bias and the hurdle induced for the rival changes by $b^\circ(V_j)V_i -$

$\min\{V_j, h^\circ\}$. By continuity, for (V_1, V_2) sufficiently close to (\tilde{V}, \tilde{V}) , this change is well approximated by $\tilde{V}b^\circ(\tilde{V}) - \min\{\tilde{V}, h^\circ\}$. When condition (23) holds, the change is positive, and so the induced hurdle jumps upward for valuation pair near (\tilde{V}, \tilde{V}) . Because the jump is proportional to V_i , the two players experience the instrument switch differently even when facing the same rival valuation close to \tilde{V} .

The no-deviation condition requires the induced hurdle to clear the threshold $\max\{\eta^H(V_j), \eta^B(V_j)\}$. Figure 4 considers a hypothetical case $V_1 = V_2 = V$ with $k = 0.5$ and plots the induced hurdles and no-deviation thresholds against V in a neighborhood of \tilde{V} . The black curve plots $V + \min\{V, h^\circ\}$ for $V < \tilde{V}$ and $V(1 + b^\circ(V))$ for $V > \tilde{V}$, exhibiting a discontinuous jump at \tilde{V} from point A to point B . In the figure, when condition (23) holds, the maximum deviation threshold is $\eta^B(V)$, and this passes between A and B at \tilde{V} . This implies that there are valuation pairs (V_1, V_2) with $V_2 < \tilde{V} < V_1$, sufficiently close to (\tilde{V}, \tilde{V}) , such that player 1 maximizes his payoff by using a head start and player 2 optimally chooses bias. Just to the left of \tilde{V} , V_2 is such that the hurdle faced by player 2 is just below $\eta^B(V)$, so that player 2 can deviate from $(h_2, b_2) = (0, 0)$ in response to the head start used by the rival. Hence the reinforcement equilibrium breaks down for valuation pairs in the neighborhood of \tilde{V} . On the other hand, suppose player 1 has a V_1 just above \tilde{V} . The hurdle induced by player 2's use of the bias instrument lies above $\eta^B(V)$, precluding a profitable deviation from $(h_1, b_1) = (0, 0)$. Hence, the preemptive equilibrium exists in the neighborhood of \tilde{V} . Putting these observations together implies that, in the neighborhood of \tilde{V} , there can be valuation pairs for which a preemptive equilibrium exists, but a reinforcement one does not.

This property depends on the no-deviation threshold $\max\{\eta^H(V_j), \eta^B(V_j)\}$ crossing in between A and B at \tilde{V} . If this falls below A , then both types of equilibrium exist for valuations pairs around \tilde{V} , since neither player has a profitable deviation from $(0, 0)$. If the no-deviation threshold crosses above B at \tilde{V} , then both players earn by deviating from the no-investment situation, and neither the reinforcement nor the preemptive equilibria will exist.

In a single-instrument environment, there is no such jump and the induced hurdle varies continuously and monotonically in the investor's valuation. The standard dominance argument then applies: if the weak player's hurdle clears $\bar{\eta}$, the strong player's hurdle—being weakly larger—does so as well. Condition (23) makes precise when the jump is large enough to overturn this logic, transforming the pre-contest

investment problem from one of pure strength into one of strategic instrument design.

The quantities in (23)— \tilde{V} , h^\diamond , $b^\diamond(\tilde{V})$, and $\bar{\eta}$ —are all determined by the cost function $c(\cdot)$ alone, so the condition is verifiable from primitives. A cost function that generates a large deterrence threshold $\bar{\eta}$ relative to $\tilde{V} + \min\{\tilde{V}, h^\diamond\}$ but small relative to $\tilde{V}(1 + b^\diamond(\tilde{V}))$ will produce a wide neighborhood of valuation pairs for which monotonicity fails, while a very convex cost function compresses this region by making all investment levels small.

6 Conclusion

We have studied a two-period contest in which rivals can prepare for a subsequent all-pay auction by investing in either an additive advantage, modeled as a head start, or a multiplicative advantage, modeled as a bias. The analysis shows that allowing multiple forms of pre-contest investment changes equilibrium behavior in a fundamental way. In any pure-strategy subgame-perfect equilibrium, at most one player invests, and the investor uses at most one instrument. A central result is that the investor's instrument choice is governed entirely by the rival's valuation. Against sufficiently weak rivals, the investor chooses a head start, while stronger rival valuations eventually induce a switch to bias. Equilibrium existence depends on whether the chosen investment leaves the rival with no profitable deviation through either investment margin. When this condition fails, no pure-strategy equilibrium exists. The model therefore highlights that pre-contest investment is not only a question of how much to spend, but also of how investment is directed in order to influence the opponent's incentives to respond.

The broader implication is that expanding the set of strategic instruments can break the monotonicity logic known from one-instrument environments. In such environments, if equilibrium exists when the low-value player invests, then equilibrium also exists when the high-value player invests. Once both head starts and bias are available, this implication need not survive. The reason is that the investor must choose an action that leaves the opponent with no profitable deviation along either margin, and this changes equilibrium existence in a fundamental way. Our quadratic-cost example shows that this possibility is not merely theoretical: there are value combinations that support preemptive equilibria, in which the low-value player invests, but not reinforcement equilibria, in which the high-value player invests.

The model also has implications for the contest stage itself. Pre-contest investment affects whether the subsequent contest is actively fought and, when it is, how intense competition becomes. A sufficiently large head start can eliminate effort in the contest stage altogether, whereas smaller advantages leave room for active competition. Expected effort therefore depends not only on prize valuations and contest technology, but also on the type, size, and identity of pre-contest investment. Analyses that abstract from endogenous preparation may therefore miss an important determinant of observed contest intensity.

The characterization of pure-strategy equilibria in this paper leaves open the nature of mixed-strategy equilibria in non-existence regions. In the single-instrument head-start model, Clark et al. (2026) show that mixed-strategy equilibria exist in which both players randomize over the decision to invest, with the mixing probabilities determined by indifference conditions on investment payoffs. The two-instrument setting introduces an additional dimension of mixing—over instrument choice as well as investment level—and characterizing the resulting equilibria is a natural extension. Several further extensions would be of interest. One is to allow contestants to differ in investment costs or in their effectiveness at generating head starts and bias. Another is to study environments with more than two players, where pre-contest investment may interact with exclusion, coalition formation, or selective targeting of rivals. A further avenue of inquiry relates to contest design, and how a principal may affect outcomes.

More generally, the framework suggests that strategic preparation deserves a more central place in contest theory, especially when contestants can choose among qualitatively different ways of improving their position before the contest begins.

Appendix

Proof of Lemma 1. Write $v_i := (1 + b_i)x_i$, and $\delta := h_i - h_j$. Then the period-2 contest is an all-pay auction in (v_1, v_2) with payoffs

$$\pi_i(v_i) = \Pr[v_i + \delta > v_j] V_i - \frac{v_i}{1 + b_i}.$$

Multiplying player i 's payoff by the positive constant $(1 + b_i)$ does not affect best responses. Hence the transformed game is strategically equivalent to

$$\begin{aligned}(1 + b_i)\pi_i(v_i) &= \Pr[v_i + \delta > v_j] (1 + b_i)V_i - v_i, \\ (1 + b_j)\pi_j(v_j) &= \Pr[v_j > v_i + \delta] (1 + b_j)V_j - v_j.\end{aligned}$$

After normalization, this is a standard complete-information head-start all-pay auction with unit bidding cost, head start δ , and effective valuations $A := (1 + b_i)V_i$, $B := (1 + b_j)V_j$. By the maintained assumption in the lemma, $A \geq B$. Also note that $\eta_i = h_j + B$, and $\eta_j = h_i + A$, which give $B = \eta_i - h_j$, $A = \eta_j - h_i$, $B - \delta = \eta_i - h_i$, and $A + \delta = \eta_j - h_j$.

Step 0 (domination - Cases 1 and 5). If $\delta \geq B$ (equivalently $h_i \geq \eta_i$), then $h_i \geq h_j + (1 + b_j)V_j \geq h_j + (1 + b_j)x_j$ for all $x_j \in [0, V_j]$, so player i wins for sure. The unique equilibrium is $x_i = x_j = 0$, with $(Eu_i, Eu_j) = (V_i, 0)$.

Symmetrically, if $\delta \leq -A$ (equivalently $h_j \geq \eta_j$), then $x_i = x_j = 0$ and $(Eu_i, Eu_j) = (0, V_j)$.

Henceforth assume $-A < \delta < B$, so that neither player is dominant.

Step 1 (structure of the mixed equilibrium).

In the asymmetric head-start all-pay auction, the mixed equilibrium may include an atom at the lower support endpoint for the advantaged player. The standard head-start all-pay characterization therefore applies, with (i) unique mixed equilibrium, (ii) affine CDFs on the continuous parts of supports, and (iii) possibly a lower-end atom. Details can be found in Clark and Riis (1995), and the equilibrium is repeated in Appendix A1 in Clark and Nilssen (2021).

Step 2 (continuation payoffs). Applying the standard head-start all-pay characterization to the normalized transformed contest (effective valuations A, B and head start δ), and then dividing back by $(1 + b_i)$ and $(1 + b_j)$, yields the continuation payoffs.

Case 2: $h_j \leq h_i < \eta_i$. Equivalently, $0 \leq \delta < B$ and $\eta_i \leq \eta_j$. Then

$$Eu_i = V_i - \frac{B - \delta}{1 + b_i} = \frac{A - (B - \delta)}{1 + b_i} = \frac{\eta_j - \eta_i}{1 + b_i}, \quad Eu_j = 0.$$

Case 3: $\eta_i < \eta_j$ and $h_i < h_j$. Equivalently, $-A < \delta < 0$. Then

$$Eu_i = V_i - \frac{B - \delta}{1 + b_i} = \frac{\eta_j - \eta_i}{1 + b_i}, \quad Eu_j = 0.$$

Case 4: $h_i < h_j \leq \eta_j < \eta_i$. Relabeling players in the previous argument gives

$$Eu_i = 0, \quad Eu_j = \frac{\eta_i - \eta_j}{1 + b_j}.$$

These are the payoff expressions in the lemma.

Step 3 (expected efforts). The same transformed-game characterization gives the first moments of the equilibrium mixed strategies in v -space. Mapping back by $x_i = v_i/(1 + b_i)$ yields the expected efforts.

Case 2: $h_j \leq h_i < \eta_i$.

$$Ev_i = \frac{(B - \delta)^2}{2B}, \quad Ev_j = \frac{(B - \delta)(B + \delta)}{2A}.$$

Hence

$$Ex_i = \frac{Ev_i}{1 + b_i} = \frac{(\eta_i - h_i)^2}{2(1 + b_i)(\eta_i - h_j)}, \quad Ex_j = \frac{Ev_j}{1 + b_j} = \frac{(\eta_i - h_i)(\eta_i + h_i - 2h_j)}{2(1 + b_j)(\eta_j - h_i)}.$$

Case 3: $\eta_i < \eta_j$ and $h_i < h_j$.

$$Ev_i = \frac{B - 2\delta}{2}, \quad Ev_j = \frac{B^2}{2A}.$$

Therefore

$$Ex_i = \frac{Ev_i}{1 + b_i} = \frac{\eta_i - 2h_i + h_j}{2(1 + b_i)}, \quad Ex_j = \frac{Ev_j}{1 + b_j} = \frac{(\eta_i - h_j)^2}{2(1 + b_j)(\eta_j - h_i)}.$$

Case 4: $h_i < h_j \leq \eta_j < \eta_i$. Relabeling gives

$$Ex_i = \frac{(\eta_j - h_j)(\eta_j + h_j - 2h_i)}{2(1 + b_i)(\eta_i - h_j)}, \quad Ex_j = \frac{(\eta_j - h_j)^2}{2(1 + b_j)(\eta_j - h_i)}.$$

Step 4 (uniqueness). Uniqueness follows from the standard head-start all-pay characterization in the normalized transformed contest (which pins down the lower-end atom, continuous supports, and affine densities uniquely).

Combining Steps 0–4 establishes the lemma. \square

Proof of Lemma 3. We derive the best response for player 1; player 2's is symmetric. From (2), W_1 is strictly decreasing in (h_1, b_1) when $h_1 \geq \eta_1$, any such choice is dominated by $(\eta_1, 0)$. Likewise, when $h_1 + (1 + b_1)V_1 \leq \eta_1$, any choice of (h_1, b_1) is dominated $(0, 0)$. In the remaining region $h_1 \in [0, \eta_1)$, $h_1 + (1 + b_1)V_1 > \eta_1$, player 1's payoff is

$$Z(h_1, b_1) := V_1 - \frac{\eta_1 - h_1}{1 + b_1} - c(h_1 + b_1).$$

For any $h_1, b_1 > 0$, a direct calculation gives

$$Z(h_1 + \epsilon, b_1 - \epsilon) - Z(h_1, b_1) = \frac{\epsilon(h_1 + b_1 - \eta_1 + 1)}{(1 + b_1)(1 + b_1 - \epsilon)}, \quad (\text{A.1})$$

which is positive when $h_1 + b_1 > \eta_1 - 1$, negative when $h_1 + b_1 < \eta_1 - 1$, and zero when $h_1 + b_1 = \eta_1 - 1$. Hence the optimizer shifts all investment to h_1 when $h_1 + b_1 > \eta_1 - 1$ and to b_1 when $h_1 + b_1 < \eta_1 - 1$, so no interior split is ever optimal. \square

Proof of Lemma 4. Case $V_1 < 1$. Feasibility for bias requires $b_1 \in [\max\{0, \frac{\eta_1}{V_1} - 1\}, \max\{0, \eta_1 - 1\}]$. Since $V_1 < 1$ implies $\frac{\eta_1}{V_1} - 1 > \eta_1 - 1$, the feasible set is empty, so player 1's problem reduces to

$$h_1^* = \arg \max_{h_1 \in [\max\{0, \eta_1 - V_1\}, \eta_1]} [V_1 - \eta_1 + h_1 - c(h_1)].$$

The best response is $(h_1^*, 0)$ if $W_1^H(V_1, \eta_1) > 0$, and $(0, 0)$ otherwise.

Case $V_1 \geq 1$. Since $\eta_1 - 1 \geq \eta_1 - V_1$, feasible choices may lie on either side of $\eta_1 - 1$. By the reallocation result (A.1), it suffices to compare the two corner problems:

$$h_1^* = \arg \max_{h_1 \in [\max\{0, \eta_1 - 1\}, \eta_1]} [V_1 - \eta_1 + h_1 - c(h_1)];$$

$$b_1^* = \arg \max_{b_1 \in [\max\{0, \frac{\eta_1}{V_1} - 1\}, \max\{0, \eta_1 - 1\}]} \left[V_1 - \frac{\eta_1}{1 + b_1} - c(b_1) \right].$$

Comparing $W_1^H(V_1, \eta_1)$ and $W_1^B(V_1, \eta_1)$ against zero yields the four subcases in Lemma 4. The argument for player 2 is symmetric. \square

Proof of Lemma 5. Each single-instrument problem is maximized over a compact interval. The head-start objective has derivative $1 - c'(h)$ and is strictly concave, so $h_i^* = \min\{U_i^H, \max\{L_i^H, h^\circ\}\}$. The bias objective has interior first-order condition

$(1+b)^2c'(b) = \eta_i$; since the left-hand side is strictly increasing, the solution is unique and $b_i^* = \min\{U_i^B, \max\{L_i^B, b^\circ(\eta_i)\}\}$.

When the lower feasibility bound depends on V_i , the objective is negative there (since $W_i(V_i, V_j, V_j - V_i, 0) < 0$ and $W_i(V_i, V_j, 0, V_j/V_i - 1) < 0$), so the optimum cannot occur at that bound. Hence $h_i^* = \min\{V_j, \max\{V_j - 1, h^\circ\}\}$ and $b_i^* = \min\{\max\{0, V_j - 1\}, b^\circ(V_j)\}$.

To locate \hat{b} : since $(1+b)^2c'(b)$ is strictly increasing, $V_j - 1 \leq b^\circ(V_j)$ iff $V_j c'(V_j - 1) \leq 1$. The function $bc'(b - 1)$ is strictly increasing with value zero at $b = 1$, so there is a unique $\hat{b} > 1$ solving $bc'(b - 1) = 1$, and $V_j c'(V_j - 1) \leq 1$ iff $V_j \leq \hat{b}$. Moreover, $\hat{b} < 1 + h^\circ \iff bc'(b - 1)|_{b=\hat{b}} < bc'(b - 1)|_{b=1+h^\circ} \iff 1 < 1 + h^\circ$, which is always true.

Monotonicity of h_i^* in V_j is immediate from its piecewise form. For b_i^* , implicit differentiation of $(1+b^\circ)^2c'(b^\circ) = V_j$ gives $db^\circ/dV_j = [2(1+b^\circ)c'(b^\circ) + (1+b^\circ)^2c''(b^\circ)]^{-1} > 0$. Independence of the value-function difference from V_i follows because all interior maximizers depend only on V_j . \square

Proof of Lemma 6. The first statement follows from the definition of $\Delta(V_j)$.

Claim : $\Delta > 0$ for $V_j \leq \hat{b}$. *Proof of Claim*: For $V_j < 1$: $b_i^* = 0$ and $h_i^* = \min\{V_j, h^\circ\}$, so $\Delta(V_j) = h_i^* - c(h_i^*) > 0$. For $1 \leq V_j \leq \hat{b}$: $b_i^* = V_j - 1$, giving $\Delta(V_j) = [h_i^* - c(h_i^*)] - [(V_j - 1) - c(V_j - 1)]$. Since $V_j - 1 \leq \hat{b} - 1 < h^\circ$, the function $h - c(h)$ is increasing at both arguments, and $h_i^* = \min\{V_j, h^\circ\} \geq V_j - 1$, so $\Delta(V_j) > 0$. This proves the claim.

Claim : Δ is strictly decreasing for $V_j > \hat{b}$. *Proof of Claim*: For $\hat{b} < V_j < 1 + h^\circ$: both h° and $b^\circ(V_j)$ are interior optimizers, so, by the envelope theorem, $d\Delta/dV_j = -1 + 1/(1 + b^\circ(V_j)) < 0$. For $V_j \geq 1 + h^\circ$: $h_i^* = V_j - 1$ is a boundary optimum, and again by the envelope theorem, $d\Delta/dV_j = -c'(V_j - 1) + 1/(1 + b^\circ(V_j)) < 0$, since $c'(V_j - 1) > 1$ for $V_j > 1 + h^\circ$. This proves the claim.

Hence, Δ crosses zero at most once for $V_j > 0$. To confirm it does cross, note $\Delta|_{V_j=1+h^\circ} = W_i(\cdot, V_j - 1) - W_i(\cdot, b^\circ(V_j)) < 0$, since $b^\circ(V_j)$ is the interior maximizer and $V_j - 1 < b^\circ(V_j)$ at $V_j = 1 + h^\circ$. Therefore $\tilde{V} \in (\hat{b}, 1 + h^\circ)$. \square

Proof of Lemma 7. $\eta_i = V_j$ in an equilibrium with i as the sole investor.

If $V_i < 1$, then $h_i = V_j - 1$ yields payoff $W_i(V_i, V_j, V_j - 1, 0) = V_i - 1 - c(V_j - 1) < 0$, so this cannot be an equilibrium strategy.

Now, suppose $V_i \geq 1$.

If player i sets $h_i^* = V_j - 1$, then Lemma 5 implies $V_j > 1 + h^\diamond$, and Lemma 6 implies $1 + h^\diamond > \tilde{V}$. Thus, $V_j > \tilde{V}$, which gives $\Delta(V_j) < 0$ (by Lemma 6), contradicting the use of head start.

If player i sets $b_i^* = V_j - 1$, then Lemma 5 implies $V_j \leq \hat{b}$, and Lemma 6 implies $\hat{b} < \tilde{V}$. Thus, $V_j < \tilde{V}$, which gives $\Delta(V_j) > 0$, contradicting the use of bias. \square

Proof of Lemma 8. *Claim:* $D^H(\eta, V_j)$ and $D^B(\eta, V_j)$ are weakly decreasing in η .

Proof of Claim. For D^H : given any feasible h'' under $\eta'' > \eta'$, set $s := \eta'' - h''$, so $0 \leq s \leq \min\{V_j, 1\}$. If $s \leq \eta'$, then $h' := \eta' - s$ is feasible under η' and yields a weakly higher payoff since c is increasing. If $s > \eta'$, the deviation payoff under η'' is $V_j - s - c(h'') \leq V_j - s < V_j - \eta'$, which is dominated by $h = 0$ under η' . Hence $D^H(\eta'', V_j) \leq D^H(\eta', V_j)$ under $\eta'' > \eta'$. The argument for D^B is identical with $q := \eta''/(1 + b'')$ in place of s , using feasibility $1 \leq q \leq V_j$ throughout. This proves the claim.

Since D^H and D^B are weakly decreasing in η , the no-deviation sets $\{\eta \geq 0 : D^H(\eta, V_j) \leq 0\}$ and $\{\eta \geq 0 : D^B(\eta, V_j) \leq 0\}$ are upper sets, and the thresholds

$$\eta^H(V_j) = \inf\{\eta \geq 0 \mid D^H(\eta, V_j) \leq 0\}, \quad \eta^B(V_j) = \inf\{\eta \geq 0 \mid D^B(\eta, V_j) \leq 0\}$$

are well defined.

Deriving $\eta^B(V_j)$. If $V_j < 1$, then the bias feasible set is empty, and $\eta^B = 0$. For $V_j \geq 1$, let $\Phi_j^B(b) := V_j - \eta_j / (1 + b) - c(b)$ denote player j 's payoff from investing b given hurdle η_j . Define $f(b) := (1 + b)^2 c'(b)$ and $g(b) := (1 + b)(V_j - c(b))$; the unconstrained optimum $b^\diamond(\eta_j)$ solves $f(b) = \eta_j$, and b^\dagger is the unique solution of $f(b) = g(b)$. Since $g'(b) = 0 \Rightarrow g(b) = f(b)$, g is maximized at b^\dagger , so $\eta_j \geq g(b^\dagger)$ renders all bias deviations unprofitable.

For $\eta_j < g(b^\dagger)$, there are two constrained cases. **(i)** If $U_j^B < b^\dagger$, then $\eta_j < 1 + b^\dagger$, and $\Phi_j^B(U_j^B) \leq 0$ requires $\eta_j \geq 1 + c^{-1}(V_j - 1)$. Since $b^\dagger > c^{-1}(V_j - 1)$ implies $g(b^\dagger) < 1 + b^\dagger$, deviation is blocked for $\eta_j \in [1 + c^{-1}(V_j - 1), g(b^\dagger))$ whenever $b^\dagger > c^{-1}(V_j - 1)$. **(ii)** If $L_j^B > b^\dagger$, then $\eta_j > (1 + b^\dagger)V_j$ is required, which contradicts $\eta_j < g(b^\dagger) = (1 + b^\dagger)(V_j - c(b^\dagger))$, so this case yields no additional no-deviation region.

The case boundary $b^\dagger(V_j) \leq c^{-1}(V_j - 1)$ reduces, via (12), (18), and the monotonicity of $(1 + x)c'(x)$, to $1 + c(\hat{b} - 1) \leq V_j$. Combining gives (17).

Deriving $\eta^H(V_j)$. Let $\Phi_j^H(h) := V_j - \eta_j + h - c(h)$ denote player j 's payoff from investing h , given hurdle η_j . The unconstrained deterrence condition $\Phi_j^H(h^\diamond) \leq 0$ gives

$\eta_j \geq V_j + h^\diamond - c(h^\diamond)$. For η_j below this, two constrained cases arise. **(i)** If $U_j^H < h^\diamond$ (i.e., $\eta_j < h^\diamond$), then $\Phi_j^H(\eta_j) \leq 0$ requires $\eta_j \geq c^{-1}(V_j)$. When $V_j < c(h^\diamond)$, optimality of h^\diamond gives $V_j + h^\diamond - c(h^\diamond) \geq c^{-1}(V_j)$, so deviation is blocked for all $\eta_j \geq c^{-1}(V_j)$. **(ii)** If $L_j^H > h^\diamond$, then the requirement is $V_j > 1 + c(h^\diamond)$, and $\Phi_j^H(L_j^H) \leq 0$ then gives the threshold $1 + c^{-1}(V_j - 1)$. Combining gives (16).

Observe that player j has no profitable deviation if and only if $\eta_j \geq \max\{\eta^H(V_j), \eta^B(V_j)\}$. Monotonicity of $\eta^H(V_j)$ and $\eta^B(V_j)$ in V_j follows from two observations: each deviation payoff is increasing in V_j for fixed η_j (by Berge's maximum theorem, D^H and D^B are continuous in V_j , and each feasible deviation payoff is strictly increasing in V_j), so the sets $\{\eta : D^H(\eta, V_j) \leq 0\}$ and $\{\eta : D^B(\eta, V_j) \leq 0\}$ expand as V_j falls, pushing both infima upward. Continuity of the thresholds follows from verification that the piecewise branches coincide at their boundary values; η^B is an exception at $V_j = 1$, where the bias feasible set switches from empty to non-empty, generating a jump discontinuity. \square

Proof of Proposition 1. Lemma 2 rules out simultaneous investment, and $(h, b) = (0, 0)$ for both players is ruled out since any player can profitably deviate to a small head start (since $h - c(h) > 0$ near zero as $c'(0) = 0$). Hence, every pure-strategy SPNE has exactly one investor.

(a). With player i as sole investor, $\eta_i = V_j$. The conditions $\max\{W_i^H, W_i^B\} > 0$ (profitability, Lemma 5) and $\eta_j \geq \max\{\eta^H(V_j), \eta^B(V_j)\}$ (no deviation, Lemma 8) are jointly necessary and sufficient for mutual best responses.

(b). The single-instrument property (Lemma 3) and independence of $\Delta(V_j) = W_i^H - W_i^B$ from V_i (Lemma 5) reduce instrument choice to a cutoff rule. Lemma 6 gives a unique zero $\tilde{V} \in (\hat{b}, 1 + h^\diamond)$ with $\Delta > 0$ for $V_j < \tilde{V}$ and $\Delta < 0$ for $V_j > \tilde{V}$. The investment levels (19) then follow from Lemmas 5 and 7.

(c). Immediate from Lemma 1: $h_i = V_j$ triggers Case 1 (zero effort); otherwise, Case 2 applies with positive expected effort. \square

Proof of Corollary 1. Part A. For $\frac{1}{2k} \leq V_2 < 1$: by (21)–(22), $\eta_2^H = V_2 + \frac{1}{2k}$ and $\eta_2^B = 0$. Since $V_2 < \tilde{V}(k)$, player 1 uses a head start with $W_1(V_1, V_2, h_1^*, 0) > 0$, inducing

$$\eta_2 = V_1 + \min\left\{V_2, \frac{1}{k}\right\}, \quad (\text{A.2})$$

which exceeds η_2^H for all $V_1 > V_2$. By Proposition 1, the reinforcement equilibrium exists with $(h_1, b_1) = (\min\{V_2, \frac{1}{k}\}, 0)$.

For $V_2 < \min\{\frac{1}{2k}, 1\}$: $\eta_2^H = \sqrt{\frac{2V_2}{k}}$, and $h_1^* = V_2$, so $\eta_2 = V_1 + V_2$, and $W_1(V_1, V_2, V_2, 0) > 0$. Therefore, a reinforcement equilibrium exists if and only if $\eta_2 \geq \eta_2^H$, i.e. $V_1 \geq \phi_1(V_2, k)$. The non-existence condition $V_1 < \phi_1(V_2, k)$ is equivalent to $k < \frac{2V_2}{(V_1 + V_2)^2}$. Since $\sup_{0 < V_2 < V_1} \frac{2V_2}{(V_1 + V_2)^2} = \frac{1}{2V_1} \rightarrow \infty$ as $V_1 \rightarrow 0^+$, the non-existence region is non-empty for every $k > 0$ and shrinks as k increases.

Part B. Player 1 uses a head start for $V_2 \leq \tilde{V}(k)$ and bias for $V_2 > \tilde{V}(k)$ (Lemma 6).

Head-start region $1 \leq V_2 \leq \tilde{V}(k)$: $W_1(V_1, V_2, h_1^*, 0) > 0$ and the induced hurdle is (A.2), so the existence condition reduces to $V_1 \geq \phi_2(V_2, k)$. That ϕ_2 may be slack for all feasible (V_1, V_2) is illustrated by $k = 1$ in Figure 1(ii).

Bias region $V_2 > \tilde{V}(k)$: player 1 sets $b_1 = b^\circ(V_2)$, earning $W_1(V_1, V_2, 0, b^\circ(V_2)) > 0$, inducing $\eta_2 = (1 + b^\circ(V_2))V_1$; the existence condition reduces to $V_1 \geq \phi_3(V_2, k)$. For $V_2 > 1 + c(h^\circ)$, $\eta_2^B > \eta_2^H$, so existence requires

$$V_1 \geq \varsigma(V_2) := \frac{kb^\dagger(V_2)(1 + b^\dagger(V_2))^2}{(1 + b^\circ(V_2))}.$$

To show that region III is non-empty for large V_2 , we establish $\varsigma(V_2) \sim c_1 V_2^{7/6}$ for some positive constant c_1 . From $kb(1+b)^2 = V_2$, $b^\circ \sim (\frac{V_2}{k})^{1/3}$; from $kb(1+b) = V_2 - \frac{kb^2}{2}$, $b^\dagger \sim (\frac{V_2}{k})^{1/2}$. Hence $kb^\dagger(1+b^\dagger)^2 \sim k(b^\dagger)^3 \sim (2V_2)^{3/2}/(3(3k)^{1/2})$ and $1 + b^\circ \sim (V_2/k)^{1/3}$, giving $\varsigma(V_2) \sim c_1 V_2^{7/6}$. Since $7/6 > 1$, $\varsigma(V_2) > V_2$ for large V_2 , so $V_1 \in (V_2, \varsigma(V_2))$ admits no reinforcement equilibrium. \square

Proof of Corollary 2. Part A. Suppose $\frac{1}{2k} < 1$, and consider $\frac{1}{2k} \leq V_1 < 1$. Player 2's optimal head start $h_2 = \min\{V_1, 1/k\}$ induces an effective hurdle $\eta_1 = V_2 + \min\{V_1, 1/k\}$ for player 1, whereas player 1's most profitable deviation arises only through a head start, yielding a deterrence threshold $\eta_1^H = V_1 + 1/(2k)$. Therefore, the deterrence condition reduces to $V_2 + \min\{V_1, 1/k\} \geq V_1 + 1/(2k)$, or, equivalently, $V_2 \geq \max\{\frac{1}{2k}, V_1 - \frac{1}{2k}\}$.

Next, consider $V_1 < \min\{\frac{1}{2k}, 1\}$. Then, the deterrence condition becomes $V_2 \geq \phi_1(V_1, k) = \sqrt{2V_1/k} - V_1$. Since $\phi_1(V_1, k)$ is increasing for $V_1 < 1/(2k)$, its minimum over the relevant range $V_1 > V_2$ is attained as $V_1 \downarrow V_2$, that is, at $\phi_1(V_2, k)$. But $\phi_1(V_2, k) > V_2$ whenever $V_2 < 1/(2k)$. Hence, in this case the deterrence condition cannot be satisfied.

It is straightforward to see that the non-existence region is non-empty, since the equilibrium ceases to exist for any $V_2 < \frac{1}{2k}$.

Part B. The proof of Part B follows the same steps used to establish the existence of the reinforcement equilibrium in Part B of Corollary 2, and is therefore omitted. The corresponding non-existence regions are non-empty because $\phi_2(V_1, k)$ and $\phi_3(V_1, k)$ are increasing in V_1 for sufficiently large V_1 . Consequently, as V_1 increases, the existence condition eventually fails to hold. \square

Proof of Proposition 2. Define $\psi_1(V) := \max\{\eta^H(V), \eta^B(V)\} - \min\{V, h^\diamond\}$ and $\psi_2(V) := \frac{\max\{\eta^H(V), \eta^B(V)\}}{1+b^\diamond(V)}$. Both functions are continuous in V , and by (23), $\psi_2(\tilde{V}) < \tilde{V} < \psi_1(\tilde{V})$. Consider the continuous functions

$$\Psi_1(V_1, V_2) := \psi_1(V_2) - V_1 \quad \text{and} \quad \Psi_2(V_1, V_2) := V_2 - \psi_2(V_1).$$

At $(V_1, V_2) = (\tilde{V}, \tilde{V})$,

$$\Psi_1(\tilde{V}, \tilde{V}) = \psi_1(\tilde{V}) - \tilde{V} > 0 \quad \text{and} \quad \Psi_2(\tilde{V}, \tilde{V}) = \tilde{V} - \psi_2(\tilde{V}) > 0.$$

By continuity, there exists an open neighborhood Θ of (\tilde{V}, \tilde{V}) on which both Ψ_1 and Ψ_2 remain positive, so that for all $(V_1, V_2) \in \Theta$,

$$V_1 < \psi_1(V_2) \quad \text{and} \quad V_2 > \psi_2(V_1).$$

Furthermore, since $W_2^B(\tilde{V}, \tilde{V}) = W_2^H(\tilde{V}, \tilde{V}) = \min\{\tilde{V}, h^\diamond\} - c(\min\{\tilde{V}, h^\diamond\}) > 0$, there exists an open neighborhood Ω of (\tilde{V}, \tilde{V}) on which $W_2^B(V_1, V_2) > 0$.

Choose $(V_1, V_2) \in \Theta \cap \Omega$ with $V_2 < \tilde{V} < V_1$. Note that $V_2 > \psi_2(V_1)$ implies $\eta_1 = (1 + b^\diamond(V_1))V_2 > \max\{\eta^H(V_1), \eta^B(V_1)\}$. This, together with $W_2^B(V_1, V_2) > 0$, implies that a preemptive equilibrium exists in which player 2 invests in bias (since $V_1 > \tilde{V}$).

Moreover, $V_1 < \psi_1(V_2)$ implies $\eta_2 = V_1 + \min\{V_2, h^\diamond\} < \max\{\eta^H(V_2), \eta^B(V_2)\}$, and therefore a reinforcement equilibrium does not exist. \square

Proposition A.1. *Fix $V_1 > V_2 > 0$. Suppose that players have access to only one instrument, either h or b , but not both; costs $c(\cdot)$ are strictly increasing and strictly convex. If there exists a subgame-perfect equilibrium in which only player 2 invests in that instrument (a preemptive equilibrium), then there also exists a subgame-perfect*

equilibrium in which only player 1 invests (a reinforcement equilibrium).

Proof of Proposition A.1. The case with a head start instrument is proved in Clark et al. (2026), Proposition 3. Here, we show the proof for bias as the singular instrument.

Feasibility requires $0 \leq b_i \leq V_j - 1$, and an investment b_i by player i induces the opponent's hurdle $\eta_j = (1 + b_i)V_i$. Recall that for any hurdle level $\eta \geq 0$, the opponent's best deviation payoff is $D^B(\eta, V_j) = \max_{b \geq 0} \{V_j - \frac{\eta}{1+b} - c(b)\}$, which gives the no-deviation from bias threshold as $\eta^B(V_j) = \inf\{\eta \geq 0 : D^B(\eta, V_j) \leq 0\}$

Since, for fixed b , the inner expression is affine in V_j and decreasing in η , D^B is increasing in V_j and decreasing in η . Hence $\eta^B(\cdot)$ is non-decreasing.

Further, the feasibility requirement, $b_i \leq V_j - 1$, implies a common hurdle cap $\eta_j \leq (1 + b_i)V_i \leq V_j V_i$. Given a target hurdle $\eta \leq V_i V_j$, the least-cost bias for player i that attains η is $b_i(\eta) = \max\{0, \min\{\frac{\eta}{V_i} - 1, V_j - 1\}\}$, so that the payoff from going from $b_i = 0$ to implementing η is

$$\kappa_i(\eta) = V_i - \frac{V_i V_j}{\eta} - c(b_i(\eta)).$$

Considering $V_1 > V_2$, for any feasible $\eta \leq V_1 V_2$, we have $\frac{\eta}{V_1} - 1 \leq \frac{\eta}{V_2} - 1$, and, restricting attention to the feasible interval $[0, V_2 - 1]$, it follows that $b_1(\eta) \leq b_2(\eta)$. Because c is strictly increasing, $c(b_1(\eta)) \leq c(b_2(\eta))$. Therefore,

$$\kappa_1(\eta) - \kappa_2(\eta) = (V_1 - V_2) + \left[c(b_2(\eta)) - c(b_1(\eta)) \right] > 0.$$

Assume a preemptive equilibrium exists, i.e., player 2 chooses some $b_2^* \in [0, V_1 - 1]$ that induces $\eta^* = (1 + b_2^*)V_2$ and satisfies: (i) deterrence of player 1: $\eta^* \geq \eta_1^B(V_1)$; (ii) profitability: $\kappa_2(\eta^*) > 0$; and (iii) feasibility: $\eta^* \leq V_1 V_2$.

Consider player 1 implementing the same hurdle η^* using $b_1(\eta^*) \in [0, V_2 - 1]$. By monotonicity of T_B , $\eta_2^B(V_2) \leq \eta_2^B(V_1) \leq \eta^*$, so player 2 is deterred at η^* . Profitability holds since $\kappa_1(\eta^*) > \kappa_2(\eta^*) > 0$. Feasibility is unchanged ($\eta^* \leq V_1 V_2$). Thus a reinforcement equilibrium (only player 1 invests) also exists.

Note that this argument also holds for boundary cases. If $b_2^* = 0$ (so that $\eta^* = V_2$) or $b_2^* = V_1 - 1$ (so that $\eta^* = V_1 V_2$), then the same argument applies because it relies only on monotonicity, the common cap $V_1 V_2$, and the dominance inequality above.

For the second part of the proposition, namely, that the reinforcement-from-

preemption property breaks down when both instruments are available, we establish the claim by means of an example, illustrated in Figure 3. \square

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