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Investment incentives in contests

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Abstract
Competition among rivals is often modeled as a contest in which the probability of winning is increasing in own effort. We consider incentives for players to attempt to influence their winning chances by undertaking a pre-contest investment. Investment costs are private information, and the outcome is stochastic. We characterize equilibrium by a threshold of the cost parameter that determines investment, and delineate cases where all players invest and when only some do. We consider how the attractiveness of the investment prospect impacts the investment decision and expected contest effort. In particular, the probability of investment success has an interesting non-monotonic effect on investment behavior.

1 Introduction

In a contest situation, agents expend some kind of resource in order to try to win a prize. The probability of securing the prize increases in the amount an agent contributes, decreases in rivals' outlays, and all contributions are irreversibly sunk. The general setup has found wide application in modeling political and economic competition.[1] This paper looks at agents' incentives to try to increase their chances of winning by making an investment prior to the contest. A common assumption is that an a priori defined pool of contestants exert effort to attempt to win the prize. The focus is on play in the actual contest and surprisingly few researchers have considered the incentives of the players to enhance their skill or ability to compete before the contest commences. This has been pointed out previously by Fu and Lu (2009) and holds true today.

[1] For a range of models and applications, see Konrad (2009) and Vojnović (2015).

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Influencing ability to compete is undeniably an important part of actual competition as noted many years ago by Danhof (1968), and later by Lichtenberg (1988) in the case of public procurement. Münster (2007) provides examples of pre-contest investment from research and marketing/advertising contests. Schaller and Skaperdas (2020) investigate how investments can affect the strategic situation between two parties who may solve a dispute through conflict or bargaining. Their model builds on the two-armed contest of Arbatskaya and Mialon (2010) in which contestants perform two activities to increase their probability of winning; one activity may be interpreted as making contest effort more efficient.[2]

This paper seeks to give insights into the trade-offs faced by agents who consider investing to improve their chances of winning a contest for a fixed prize. A pool of initially identical agents receive private information about how much it would cost them to make an investment in acquiring a skill that could improve his chances of winning the contest. Specifically, the skill reduces the agent's marginal cost of making contest effort. In an important extension of previous work, skill acquisition is stochastic, and there is a chance that the investment fails. Agents take both the cost and the stochastic return into account when making the decision whether to invest or not. The outcome of the skill acquisition process is common knowledge at the contest stage, in which some skilled players compete against unskilled. The composition of the contestants influences the amount of resources used in the contest, and hence the final payoff; this is another factor that affects the incentive to invest in skill acquisition. An extension considers relaxing the assumption of identical players, by dividing the initial pool into skilled and unskilled. By doing this, the analysis is valid for nascent competition (with no skilled agents initially), as well as contest situations that may already be established; in the latter, it is not unreasonable to assume that some agents are already skilled.

Our results highlight how the incentive to invest and total expected contest effort are affected by the likelihood and size of the investment return. As we document below, previous research considers investment that has a deterministic, predictable effect on the ability to compete at the contest stage. Our research thus complements the scant literature on pre-contest investment, and specifically investigates how the stochastic return affects investment and efforts. Specifically we derive a threshold for the investment cost that determines investment in equilibrium; those with a privately-known cost below the threshold make the investment. We characterize two regimes that can emerge in equilibrium. In a full-investment outcome, all players invest at the pre-contest stage, and in a partial investment regime only those with a cost below the threshold will make an attempt to become skilled. As one would expect, we show that the threshold cost level is increasing when the return to investment increases, so that more agents will make the investment.

Our main focus is on the stochastic nature of the investment return, and we identify an important non-monotonicity effect that the probability of investment success has on the investment threshold. A low probability of succeeding gives a low investment incentive as one would expect. More surprising is that a high probability of success

[2] Amegashie (2012) notes that an agent may increase his own chances of winning through efforts or investments that may be destructive for the opponents' efficiency in the contest dimension.

can also deter investment. The intuition is as follows: a successful investment is most valuable at the contest stage if few others have succeeded. If many have succeeded with the investment then the contest is fought out by many strong players, and much of the contest prize is dissipated. This happens when the probability of successful investment is high. We show, however, that the amount of expected contest effort increases unambiguously in this parameter in spite of the non-monotonic effect on investment. Contest effort increases the more strong players there are at the contest stage. A high probability of success makes these states more likely, and this effect outweighs the possible negative impact through the investment incentive, increasing expected effort.

When pool of agents of fixed size includes some already skilled players, two effects can be identified in determining investment decisions. First, the unskilled are encouraged to invest in acquiring the skill since they face fewer investing rivals; however, the fact that there already are skilled players at the contest stage, reduces the expected payoff at that stage, discouraging investment. We show that the latter effect dominates. The presence of already skilled players makes it less likely that new competitors will try to acquire the skill at the investment stage. Agents looking for an investment opportunity may then be tempted to look for "another hill to climb", rather than investing in the same prospect that has attracted interest previously. This echoes the ideas in Taub (2024), who illustrates with the case of the home entertainment industry around the turn of the millennium in which an established business model (store-rented DVDs) was challenged by a mail-order start-up. Rather than investing in the model that was already dominated by some large firms (such as Blockbuster), Netflix invested in a different type of infrastructure.[3]

Konrad (2002) was the first to consider how investment may affect contest equilibrium. In his model, investment is one-sided; an incumbent can make an investment that increases the value of gaining or retaining power. There is a deterministic and continuous relationship between investment and returns, resembling a production function. The effect of investment in the models analyzed by Keskin and Sağlam (2019) and Schaller and Skaperdas (2020) can also be interpreted as a deterministic, continuous production function, translating into an improved efficiency of contest effort. Münster (2007) and Fu and Lu (2009) assume that pre-contest investments have a deterministic and continuous effect on the cost of making contest effort. Qualitatively, there is little difference between investments that affect efficiency or cost in a known manner. Our model differs from all of these since the return to investment is uncertain, achieving a reduction in the marginal cost of effort probabilistically. At the time that the investment decision is made, agents do not know how many unskilled and skilled rivals they will ultimately face in the contest.

Contest models can be broadly dichotomized according to the success function utilized. An all-pay auction belies the models of Konrad (2002) and Münster (2007) in which the player with the highest contest score is the winner.[4] A feature of the all-pay auction is that it often has an equilibrium in which a single player expects a positive payoff, whilst all others expect zero. This property is often useful in solving

[3] For more on the competition between Blockbuster and Netflix, see Dalton and Logan (2024).

[4] The contest score may be effort itself, or some affine transformation of contest effort.

contest models with several stages. Clark and Kundu (2025) exploit this to solve a contest with both endogenous entry and pre-contest investment. The current paper features a Tullock contest, in which the ratio of a contestant's score in relation to the sum of scores determines the probability of winning. In these contests, several players would typically expect to receive a non-zero payoff, and this complicates the analysis of models with several stages considerably. Keskin and Sağlam (2019) and Schaller and Skaperdas (2020) use a two-player Tullock contest together with deterministic pre-contest investment. In a Tullock contest with investment, Fu and Lu (2009) find that no equilibrium with investment exists for four or more players. Further, they show that the equilibria with investment that exist for two and three players are valid for a very limited parameter space. By considering a stochastic return on investment, we can derive equilibria in which more players invest.

There are several related strands of literature in which an action or event can change the composition of the contestants, or the rules of the game being played. One regards building up human capital in order to be better placed to compete (see for example Grossmann and Dietl 2009; Grossmann et al. 2011). Another considers the possibility that previous contest efforts may be carried over to a later competition (e.g. Baik and Lee 2000; Schmitt et al. 2004), while a third allows actual previous contest outcomes to partly determine the win probability in the current game (for example Beviá and Corchón 2013; Clark and Nilssen 2018). We do not consider these features in our analysis, however.

The paper is organized as follows. Section 2 outlines the model, which is then solved for the equilibrium in Sects. 3 and 4. Section 5 discusses how the model parameters affect the incentives to make pre-contest investments and contest effort. Section 6 concludes. Proofs that are not in the text are included in the Appendix.

2 Model

Consider $n \geq 2$ agents who compete in a contest to win a prize of value 1. Prior to the contest stage, agents can invest in acquiring a skill which affects their marginal costs of effort in the actual contest. An agent's investment cost is θ_i/k, where $\theta_i, i \in \{1, 2, ..., n\}$ are independent random variables uniformly distributed over $[0, 1]$ and $k > 0$ is a constant. The return to investment is stochastic. Upon investment, an agent acquires the skill with probability $q \in (0, 1)$. Those who do not invest do not acquire the skill. A skilled agent's marginal cost of exerting contest effort is given by $\alpha \in (0, 1)$ and an unskilled agent's marginal cost of effort is 1. The parameters k, α, q are common knowledge.

The probability of agent $i \in \{1, 2, ..., n\}$ winning the contest is given by

$$\rho_i(e_1, ..., e_n) = \frac{e_i}{\sum_{j=1}^{n} e_j},$$

where e_i denotes the effort level of agent i. The expected payoff of agent i with marginal cost $c \in \{\alpha, 1\}$ is

$$v_i = \rho_i - ce_i.$$

The game proceeds as follows:

- Stage 0: Nature chooses the type θ_i of agent $i = 1, 2, \ldots, n$, and $i's$ type is private information.
- Stage 1: Agent $i \in \{1, 2, \ldots, n\}$ decides whether to invest in skill by incurring an investment cost of θ_i/k. Nature decides whether an agent realizes the skill upon investment. Suppose that $s \leq n$ agents acquire the skill. The number of skilled agent s is public information.
- Stage 2: The agents participate in the contest to win the prize. Nature chooses the winner of the contest, using $\rho_i(e_1, ..., e_n)$.

We study the perfect Bayesian equilibrium in symmetric strategies.[5]

3 Contest stage

The analysis begins at the final stage, where the number of skilled players $s \in \{0, 1, \ldots, n\}$ is common knowledge. We consider the equilibrium in symmetric strategies such that all skilled agents choose e_S and all unskilled agents choose e_U. The equilibrium effort profile (e_S, e_U) solves

$$e_S = \arg\max_e \frac{e}{e + (s-1)\, e_S + (n-s)\, e_U} - \alpha e, \tag{1}$$

$$e_U = \arg\max_e \frac{e}{e + s e_S + (n-s-1)\, e_U} - e. \tag{2}$$

The first-order conditions, after replacing e by e_S in case of (1) and by e_U in case of (2), can be expressed as

$$\frac{T - e_S}{(T)^2} - \alpha = 0, \tag{3}$$

$$\frac{T - e_U}{(T)^2} - 1 = 0, \tag{4}$$

where $T = s e_S + (n-s)\, e_U$ is the total effort. The first-order conditions yield a boundary solution if $T \geq 1$, in which case, the unskilled contestant's marginal return from effort is always negative. Solving (3) and (4), we get

$$\begin{aligned} e_S &= T\,(1 - \alpha T)\,, \\ e_U &= T\,(1 - T)\,, \\ \text{and } T &= \frac{n-1}{n - (1-\alpha)\, s}. \end{aligned} \tag{5}$$

[5] Asymmetric equilibria can arise in endogenous-investment contests, where some players may or may not invest regardless of circumstances, while others follow a threshold-based investment strategy. We analyze the symmetric equilibrium in cost-based strategies, as it serves as a natural focal point in the absence of a clear coordination mechanism, given that investment type is private information.

The above solution characterizes the equilibrium effort only if $T < 1$, or equivalently, $s < \frac{1}{1-\alpha}$. If $s \geq \frac{1}{1-\alpha}$, then all unskilled contestants exert zero effort; we have a contest among s skilled contestants. In that case, from the first-order condition (1) and the fact that $T = se_S$, we get

$$\begin{aligned} e_S &= \frac{s-1}{\alpha s^2}, \\ T &= \frac{(s-1)}{\alpha s}. \end{aligned} \tag{6}$$

We express the aggregate effort as a function of n and s as

$$T(n,s) := \frac{n-1}{n-(1-\alpha)s}. \tag{7}$$

Then, we can write the equilibrium effort levels as

$$e_S(n,s) = \begin{cases} T(n,s)(1-\alpha T(n,s)) & \text{if } s < \frac{1}{1-\alpha} \\ T(s,s)(1-\alpha T(s,s)) & \text{if } s \geq \frac{1}{1-\alpha} \end{cases}, \tag{8}$$

$$e_U(n,s) = \begin{cases} T(n,s)(1-T(n,s)) & \text{if } s < \frac{1}{1-\alpha} \\ 0 & \text{if } s \geq \frac{1}{1-\alpha} \end{cases}. \tag{9}$$

The stage-2 expected payoffs (when investment cost is sunk) of a skilled agent and an unskilled agent, given (n,s), are

$$v_S(n,s) = \begin{cases} (1-\alpha T(n,s))^2 & \text{if } s < \frac{1}{1-\alpha} \\ (1-\alpha T(s,s))^2 & \text{if } s \geq \frac{1}{1-\alpha} \end{cases}, \tag{10}$$

$$v_U(n,s) = \begin{cases} (1-T(n,s))^2 & \text{if } s < \frac{1}{1-\alpha} \\ 0 & \text{if } s \geq \frac{1}{1-\alpha} \end{cases}, \tag{11}$$

respectively.

4 Investment

Consider stage 1 at which the investment strategy affects the number of skilled players. An agent's investment decision is contingent on his type θ. If there are $s \in \{0,1,\ldots,n-1\}$ skilled among the other $n-1$ players, a player of type θ_i will invest if the expected payoff from being skilled is at least as large as remaining unskilled:

$$q\mathbb{E}_s[v_S(n,s+1)] + (1-q)\mathbb{E}_s[v_U(n,s)] - \theta_i/k \geq \mathbb{E}_s[v_U(n,s)] \tag{12}$$

where the expectation is taken over the probability distribution of s. Simplifying (12) gives:

$$q\mathbb{E}_s[v_S(n, s+1) - v_U(n, s)] \geq \frac{\theta_i}{k}. \tag{13}$$

The relative gain from being skilled is then defined as:

$$\Delta(n, s) := v_S(n, s+1) - v_U(n, s).$$

Using (10) and (11), this is given by

$$\Delta(n, s) = v_S(n, s+1) - v_U(n, s) \tag{14}$$
$$= \begin{cases} (1 - \alpha T(n, s+1))^2 - (1 - T(n, s))^2 & \text{if } s+1 < \frac{1}{1-\alpha} \quad \text{(I)} \\ (1 - \alpha T(s+1, s+1))^2 - (1 - T(n, s))^2 & \text{if } s < \frac{1}{1-\alpha} \leq s+1 \quad \text{(II)} \\ (1 - \alpha T(s+1, s+1))^2 & \text{if } \frac{1}{1-\alpha} \leq s \quad \text{(III)} \end{cases}.$$

When considering the relative gain from acquiring the skill, a player must take account of the contest that arises if he is successful and if he fails. In the former case, there are $s+1$ skilled players, and in the latter s. Hence (14) takes three distinct forms in three different cases, each valid in a specific region of the (α, s)-space; these regions are delineated in Fig. 1. First is case I in which all players compete in the contest, even if the agent acquires the skill; the second case (II) reflects the situation in which an agent's acquisition of the skill leads to a contest between only $s+1$ skilled players, whereas not acquiring the skill results in a contest in which all s skilled and $n-s$ unskilled players participate. In the third case (III), the number of skilled players is already so large that the unskilled do not participate in the contest and receive a payoff of zero. The gain to investing in this areas is hence the payoff from participating in the contest as one of $s+1$ skilled players. The first case occurs for low numbers of skilled players, whilst the third holds for relatively many skilled players. The intermediate second case divides these two situations, and is valid for a single value of s.

Suppose for now that players follow a threshold strategy, in which there is a value θ_I such that all those with cost $\theta < \theta_I$ will invest. Under this assumption, we can rewrite (13) as

$$\xi(n, \theta_I) := q\left[\sum_{s=0}^{n-1} \binom{n-1}{s} (q\theta_I)^s (1 - q\theta_I)^{n-1-s} \Delta(n, s)\right] \geq \frac{\theta_i}{k}, \tag{15}$$

where $\xi(n, \theta_I)$ measures a player's gross expected return on investment, conditional on the other $(n-1)$ players adopting the threshold strategy with cutoff θ_I. The expression in $\xi(n, \theta_I)$ reflects that only players with types $\theta \leq \theta_I$ invest, each succeeding in acquiring the skill with probability q. Among the $(n-1)$ competitors, exactly s skilled players can be drawn in $\binom{n-1}{s}$ ways and the probability of each draw is $(q\theta_I)^s (1 - q\theta_I)^{n-1-s}$. The right-hand side of (15) represents the investment cost for a player of type θ_i and in equilibrium the indifferent (marginal) investor satisfies $\theta_i = \theta_I$.

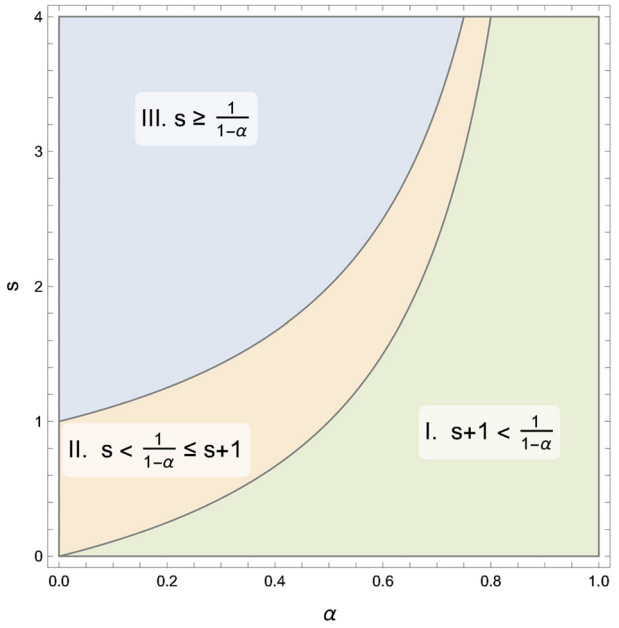

Fig. 1 Delineation of investment incentives

The following lemmas establish the existence of a unique equilibrium in threshold strategies among all cost-based symmetric investment strategies. First, Lemma 1 shows that the relative gain from skill acquisition is always positive but strictly decreasing in the number of skilled players.

Lemma 1 *Consider* $s \in \{0, 1, \ldots, n-1\}$. $\Delta(n, s) > 0$ *and is strictly decreasing in* s.

An agent that is considering investment takes account of the result in Lemma 1. There is always a benefit to acquiring the skill but it is strictly lower the more rivals that are skilled. Furthermore, Lemma 1 is important in determining how $\xi(n, \theta_I)$ changes with the threshold θ_I. To see this, we use Fact 1 that is proved in Clark and Kundu (2025, Lemma A.1):

Fact 1 *Suppose* X *follows Binomial*(m, p). *Then* $\mathbb{E}[f(X)]$ *is (strictly) decreasing in* p *if* $f(X)$ *is (strictly) decreasing in* X.

Observe that

$$\xi(n, \theta_I) = q\mathbb{E}_s[\Delta(n, s)] \text{ where } s \backsim \text{Binomial}(n-1, q\theta_I)$$

Therefore, Lemma 1 and Fact 1 together imply that $\xi(n, \theta_I)$ is strictly decreasing in θ_I for any given q. Lemma 2 documents this observation; the proof is a straightforward application of Lemma 1 and Fact 1.

Lemma 2 *For given* $q \in (0, 1)$, $\frac{\partial \xi(n,\theta_I)}{\partial \theta_I} < 0$.

A player's gross benefit from investing is determined entirely by the strategies chosen by the other $n-1$ players. A player of type θ_i will invest only if this gross benefit exceeds the investment cost θ_i/k; in equilibrium, this only gives rise to a cost-based threshold strategy. Lemma 2 shows that the gross benefit of investing is strictly decreasing in the threshold θ_I, which lets us determine the equilibrium threshold from (15). Specifically, if the investment threshold is strictly below 1, then the inequality in (15) must be binding for the the agent who is indifferent between investing and not. Proposition 1 documents these facts; the proof follows directly from the discussion above.

Proposition 1 *Agents follow a threshold strategy in equilibrium, in which there exists* $0 < \theta_I \leq 1$ *such that all agents with type* $\theta \leq \theta_I$ *invest. Moreover, if* $\theta_I < 1$*, it is the unique solution to*

$$\xi(n, \theta_I) = \frac{\theta_I}{k}. \tag{16}$$

If $\xi(n, \theta_I) > \theta_I/qk$ for all $\theta_I \leq 1$, then (16) has no solution. In this case, all types invest, and hence the investment threshold $\theta_I = 1$. Since $k\xi(n, \theta_I) - \theta_I$ is strictly decreasing in θ_I, we can identify this possibility by comparing $k\xi(n, 1)$ to 1. Specifically, if $k\xi(n, 1) \geq 1$, all types invest; otherwise, the investment threshold θ_I is uniquely determined by the solution of (16). Two possibilities may, therefore, arise in equilibrium.

- **Full investment**: All types invest: $\theta_I = 1$. This occurs if $k\xi(n, 1) \geq 1$.
- **Partial investment**: A subset of types invest: $\theta_I < 1$. This occurs if $k\xi(n, 1) < 1$.

When an agent gets to know that he is of type θ, he can work out his expected payoff given the equilibrium in Proposition 1. Denote this payoff by $v(\theta, n)$. This is at least as large as the expected payoff from being unskilled, $\mathbb{E}_s[v_U(n, s)]$. To this we add the net expected return from investing; this depends upon whether there is full or partial investment.

In case of full investment, i.e., when $k\xi(n, 1) \geq 1$, the gross benefit is $\xi(n, 1)$ and the investor pays θ/k. Therefore, the expected return from investing is given by $\xi(n, 1) - \frac{\theta}{k}$. In case of partial investment, i.e., when $k\xi(n, 1) < 1$, the gross benefit is $\xi(n, \theta_I)$, which in equilibrium coincides with θ_I/k, and the investor only pays θ/k. Therefore, the expected return from investing is given by $(\theta_I - \theta)/k$. All non-investing players simply receive the expected payoff from being unskilled, which is $\mathbb{E}_s[v_U(n, s)]$.

Proposition 2 documents the equilibrium behavior; the proof follows directly from the discussion above.

Proposition 2 *The following possibilities can arise in equilibrium.*

1. If $k\xi(n, 1) \geq 1$*, then there is full investment, i.e.,* $\theta_I = 1$*. In this case, the expected payoff of an agent of type* θ *is given by*

$$v(\theta, n) = \frac{k\xi(n, 1) - \theta}{k} + \mathbb{E}_s\left[v_U(n, s)\right], \; s \backsim \text{Binomial}(n-1, q). \tag{17}$$

2. If $k\xi\,(n,1) < 1$, then there is partial investment and the investment threshold θ_I uniquely solves (16). In this case, the expected payoff of an agent of type θ is given by

$$v\,(\theta,n) = \begin{cases} \dfrac{\theta_I - \theta}{k} + \mathbb{E}_s\left[v_U\,(n,s)\right] & \text{if } \theta \le \theta_I \\ \mathbb{E}_s\left[v_U\,(n,s)\right] & \text{if } \theta_I < \theta \le 1 \end{cases}, \; s \backsim \text{Binomial}\,(n-1, q\theta_I)\,. \tag{18}$$

In general, the distribution of the number of investors s is given by $Binomial \sim (n, q\theta_I)$.

5 Comparative static effects

Now that we have derived the equilibrium, we can discuss how the underlying parameters affect investment behavior and contest efforts. In the following sections, we discuss how the investment incentives and the total contest efforts are affected by the attractiveness of the investment prospect. We also consider a situation in which the market may already be populated by some skilled agents.

5.1 Investment return

The investment return is characterized by the size of the new marginal cost of contest effort from skill acquisition, α, and the probability of skill acquisition, q. The marginal cost α determines the comparison an agent makes between returns from different types of contests; see (14) and Fig. 1. For a fixed value of α, increasing the number of skilled players by one can change the type of contest from one that involves all agents (both skilled and unskilled) to one that includes only skilled players. When α is low, and hence the return to investing is high, this shift can occur with a relatively small number of skilled players. As α increases, the agent's incentive to invest is more likely determined by the relative payoff between becoming skilled and participating in a contest with all players, and participating in the same contest as an unskilled player.

See Fig. 1 for an illustration. Area I, delineating combinations of high α and/or low s values, corresponds to the parameter values where a contest with all players competing persists even when the number of skilled players changes by one. Conversely, in area III, only skilled players compete. Area II represents a transitional phase, where an extra skilled player shifts the contest from one in which all players compete to one in which only skilled players make contest effort.

Proposition 3 shows the precise comparative static effect that increasing α has on the investment incentive $\Delta_j(n,s)$ for each case $j = I, II, III$.

Proposition 3 $(i)\ \dfrac{\partial \Delta_I(n,s)}{\partial \alpha} \le \dfrac{\partial \Delta_{II}(n,s)}{\partial \alpha} < 0,\ (ii)\ \dfrac{\partial \Delta_{III}(n,s)}{\partial \alpha} = 0.$

When only skilled workers participate in the contest (area III), α has no effect on the investment incentive. In the other cases, reducing α makes the investment more

attractive and the temptation to invest rises. It rises the most in area I in Fig. 1, since all agents—skilled and unskilled—will compete in the contest. Then it is a big advantage to be skilled. In area II, increasing the number of skilled workers by one, means that the contest transitions from one in which skilled and unskilled play to one with just strong competitors. The strength of the competition means that more of the prize is dissipated and there is less to be gained from investing than in area I.

Next, we turn to the comparative-statics of the investment threshold. If $\theta_I = 1$, then changes in the model's parameters have no marginal effect on the threshold. Under partial investment, we can derive meaningful comparative-statics by focusing on the solution of the investment-indifference condition. To this end, let $\theta_{\text{int}}(q)$ denote the internal solution to (16), so that $\theta_I = \theta_{int}$ whenever $\theta_I < 1$.

Parts (i) and (ii) of Proposition 3, imply that $\mathbb{E}_s[\Delta(n, s)]$, and hence $\xi(n, \theta_I)$, is decreasing in α. Then it follows from (16) that θ_{int} is decreasing in α. Hence, the equilibrium investment threshold also decreases with α; only if $\theta_I = 1$ after the increase in α, will this parameter not affect the threshold. The marginal cost of investment $1/k$ also has a similar monotone effect on the investment threshold. Increasing k reduces the cost of investment without altering $\xi(n, \theta_I)$; hence, whenever $\theta_I < 1$, the equilibrium investment threshold, given by θ_{int}, increases in k. Proposition 4 documents these results; the proof follows directly from the discussion above.

Proposition 4 $(i)\ \dfrac{\partial \theta_{int}}{\partial \alpha} < 0,\ (ii)\ \dfrac{\partial \theta_{int}}{\partial k} > 0.$

Unlike α and k, the probability of skill acquisition q affects the investment threshold in a non-monotone way. The non-monotonicity arises because of the two opposing effects of q on the investment incentive. On the one hand, an increase in q makes a player more likely to acquire the skill, giving a positive effect on the incentive to invest. On the other hand, the player expects to face more skilled competitors which dampens the incentive to invest. While the first effect changes at a linear rate, the second effect does so in a non-linear way and can dominate the first effect for high values of q. The combined effect determines the investment decision. We prove the following result in the Appendix:

Proposition 5 *$\theta_{\text{int}}(q)$ is unimodal on $q \in (0, 1)$:*
1. It is either strictly increasing on $q \in (0, 1)$, or
2. There exists a unique $q^* \in (0, 1)$ such that $\theta'_{\text{int}}(q) > 0$ for $q < q^*$, $\theta'_{\text{int}}(q^*) = 0$, and $\theta'_{\text{int}}(q) < 0$ for $q > q^*$.

Proposition 5 implies that there can be three possible shapes of the $\theta_I(q)$ function, and these are given in Fig. 2. In panel A, the dotted curve illustrates part 1 of Proposition 5, in which the threshold always increases in q, never reaching $\theta_I = 1$, implying partial investment for all $q \in (0, 1)$. The whole line indicates partial investment for low q, and then full investment once θ_I reaches 1. In panel B there is a maximal value of the investment threshold that is below 1, and there is partial investment for all $q \in (0, 1)$. In panel C, there is a portion of θ_I that is 1 for intermediate values of q, and below otherwise. Hence, there is full investment for intermediate values of q in this case, and partial investment outside of this range. Panels B and C demonstrate cases in which the investment threshold in non-monotonic in the probability of success.

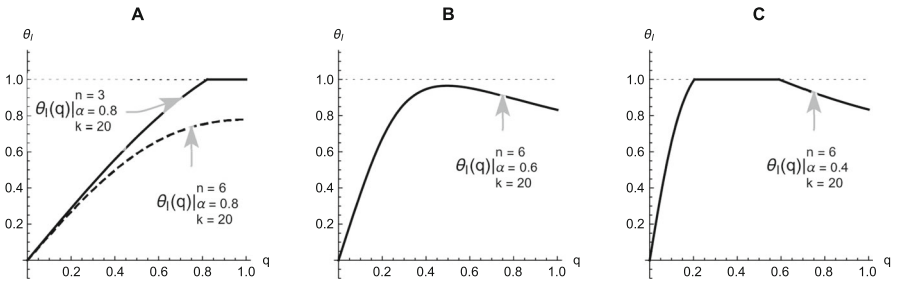

Fig. 2 Possible shapes for $\theta_I(q)$

5.2 Expected total efforts

The contest literature focuses often on the total efforts that are expended in trying to win the prize. In our case, for a given number of skilled players s in the total pool of n players, the total effort is given by

$$T = \begin{cases} T(n,s) \text{ if } s < \dfrac{1}{1-\alpha} \\ T(s,s) \text{ if } s \geq \dfrac{1}{1-\alpha} \end{cases} . \tag{19}$$

The following lemma shows that T is bounded and increasing in both s and n. The proof follows directly from the expressions for T derived in (6) and (7).

Lemma 3 *Consider $n \geq 2$ and $s \in \{0, 1, \cdots, n\}$.*

1. For given n, T is strictly increasing in s for all $s \leq n$; and $T \in \left[\frac{n-1}{n}, \frac{n-1}{\alpha n}\right]$.

2. For given s, T is increasing in n for all $n \geq s$; and $T \in [\frac{s-1}{\alpha s}, 1)$ if $\frac{s-1}{\alpha s} < 1$; and $T = \frac{s-1}{\alpha s}$ if $\frac{s-1}{\alpha s} \geq 1$.

Increasing the number of skilled players in the fixed pool always increases the total effort. Hence, the effort interval is bounded at the bottom by the case with no skilled players, and at the top when all players are skilled. Fixing s, and increasing the pool of players essentially adds an unskilled player. This increases effort only if s is sufficiently low that the unskilled newcomer will exert effort in the contest.

At the outset of the game, the expected total effort depends on the distribution of s, which follows Binomial $(n, q\theta_I)$. The distribution depends on the probability of successful investment q and the entry threshold θ_I, the latter being itself a function of q. The following proposition shows that the expected total effort increases with q. The key to establish this result is to show that $q\theta_I$ is increasing in q, even though $\theta_I(q)$ need not be monotonic. We can therefore apply Fact 1 to show that the expected total effect is increasing in q.

Proposition 6 *For any given $n \geq 2$, the expected total effort $\mathbb{E}_s[T]$ is strictly increasing in q.*

In isolation, we showed that q has a non-monotonic effect on the propensity to invest. The effect of this parameter on expected contest effort is, however, unambiguously

positive. From Proposition 3, when we reach the contest stage there is most effort if there are many skilled players. The probability that we end up in such a state is depends on the investment incentive and the probability that investors succeed. The former effect is non-monotonic in q, whilst the second increases in q. Proposition 6 shows that the second effect always dominates in cases where they pull in opposite directions.

5.3 Investment incentives in the presence of already skilled players

We extend our basic model to account for scenarios where some players already possess skills at the outset of the game, while others undergo a pre-contest investment stage. We may think of the analysis up to now as representing a contest in a nascent industry (with no skilled players initially), whereas we henceforth look at a more mature industry in which some competitors have achieved a skill level. Formally, we modify the timeline of the game as follows: At stage 0, nature chooses the type θ_i of the non-skilled agents $i \in \{1, 2, \ldots, n-m\}$ where $m \in \{0, 1, \ldots, n-2\}$ denotes the number of agents who have already acquired skill at stage 0. The non-skilled agents then proceed to the investment stage and finally at stage 2, all agents compete in a contest (although some may choose not to exert effort). As before, a non-skilled agent's return to investment is decreasing in his type and therefore, his investment strategy continues to be a threshold strategy. We let $\theta_I(m)$ denote the investment threshold in this modified game. The investment-indifference condition is given by

$$q\left[\sum_{s=0}^{n-m-1}\binom{n-m-1}{s}(q\theta_I)^s(1-q\theta_I)^{n-m-1-s}\,\Delta(n,s+m)\right]=\frac{\theta_I}{k}, \qquad (20)$$

and the investment threshold is the minimum of the solution of (20) and $\theta_I = 1$. The investment-indifference condition is derived from the fact that an investing agent receives an expected return of $\Delta(n, s+m)$ when there are exactly s skilled players among the other $(n-m-1)$ potential investors competing against the indifferent agent at the investment stage. An increase in m brings about two partial effects. As the pool of potential investors shrinks, the indifferent agent's relative success probabilities at the investment stage alter in a non-monotone way. However, his expected return at the contest stage, $\Delta(n, s+m)$, diminishes with m. Proposition 7 proves that the dampening effect dominates, resulting in a reduction of the investment threshold with m.

Proposition 7 *In the modified game, the investment threshold $\theta_I(m)$ is decreasing in m, the number of agents who have already acquired the skill at stage* 0.

Figure 3 illustrates the result, where $m = 0$ is the case examined previously. Fixing the number of players in total, the more that are already skilled, the less likely it is that unskilled contestants will make the investment. There is little incentive to attempt to catch up others, and the stochastic nature of the return exacerbates the problem. In the previous analysis, a player invests based on the expectation of how many rivals

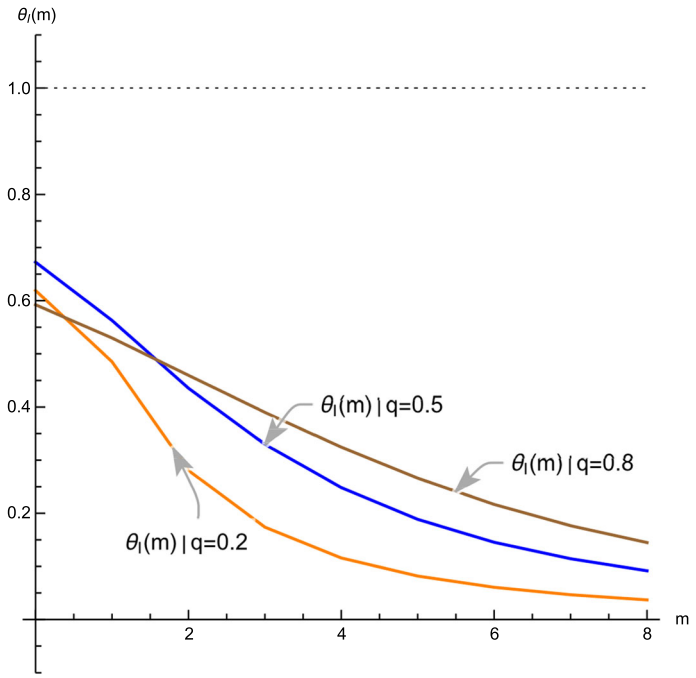

Fig. 3 $\theta_I\,(m)$ against m (with $n = 10$, $\alpha = 0.5$, and $k = 15$)

that will achieve the skill. Now a player knows that he faces a number of skilled rivals with certainty, and has to calculate the expected return based on the predicted behavior of unskilled rivals. Proposition 7 implies that we should expect less pre-contest investment when there are already established actors in the market. This can potentially have consequences for a contest designer who is interested in fostering skill investment. Suppose, for example that a National Research Council wishes to encourage investment and innovation in research institutions. It could announce a call for projects in which some institutions have already built up competence. However, encouraging new researchers would require making a call that levels the playing field in such a way that one requires a skill set that few already possess.

6 Conclusion

Few researchers have considered how pre-contest investment affects the actual contest, even though this is a prevalent feature of competition. Previous work has considered investment that has a deterministic effect on rivals' ability to compete in the contest. Our focus has been on stochastic investment returns, in which investment decisions affect the composition of skilled and unskilled rivals that contest a final prize. This in turn affects the amount of effort expended in the contest.

We characterize investment with a return and a probability of success in a model in which players' investment cost is private information. The equilibrium is characterized

by the threshold strategy on the cost parameter; agents with a cost below the threshold invest, and those above do not. This leads to two potential investment regimes in equilibrium: one in which all invest (i.e. all agents have a cost below the threshold), and one in which only some invest. The former occurs when the investment opportunity is sufficiently favorable, although not all investors will succeed in acquiring the skill. In the latter regime, there is an internal investment threshold. A novel finding is that the threshold investment cost is not monotonic in the probability of a success. When the chance of success is low, the investment is unattractive but the potential gain to achieving skill is large since one expects to meet few other skilled rivals at the contest stage. A high probability of success is favorable for realizing the skill, but one expects to meet other skilled contestants and this reduces the expected payoff for making the investment. In spite of this non-monotonicity, total expected effort at the contest stage is increasing in the probability of investment success.

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

Appendix

The Appendix contains the proofs that are not already contained in the text.

Proof of Lemma 1 Regions I, II and III refer to Fig. 1 and equation (14).

Part 1. First we show that $\Delta(n, s) > 0$.

Consider first region I in which $s + 1 < \frac{1}{1-\alpha}$, i.e. $s < \frac{\alpha}{1-\alpha}$. Then we need to show $T(n, s) \geq \alpha T(n, s + 1)$, i.e.

$$\frac{n-1}{n-(1-\alpha)s} > \frac{\alpha(n-1)}{n-(1-\alpha)(s+1)} \Rightarrow \frac{n-1}{1-\alpha} > s.$$

which must hold since $\frac{n-1}{1-\alpha} > \frac{\alpha}{1-\alpha} > s$ in this case.

Take now region II in which $s < \frac{1}{1-\alpha} \leq s+1$, equivalent to $\frac{s}{s+1} \geq \alpha > \frac{s-1}{s}$; then we must show $T(n,s) \geq \alpha T(s+1, s+1)$, i.e.

$$\frac{n-1}{n-(1-\alpha)s} > \frac{s}{s+1}. \tag{A.1}$$

The left-hand-side is decreasing in α, so the inequality is least likely to hold for high values of this parameter. Setting $\alpha = \frac{s}{s+1}$, which is the highest value of α in this region, makes (A.1) $n \geq \frac{1+2s}{1+s}$; as s grows, the right-hand-side approaches 2 from below, and since $n \geq 2$ the inequality holds.

In region III, $s \geq \frac{1}{1-\alpha}$, $\Delta(n,s) = \frac{1}{(s+1)^2} > 0$ clearly holds.

Part 2. We now show that $\Delta(n,s)$ is strictly decreasing in s.

Rewrite this expression as

$$\begin{aligned}\Delta(n,s) &= v_S(n,s+1) - v_U(n,s) \\ &= \begin{cases} (1-\alpha T(n,s+1))^2 - (1-T(n,s))^2 & \text{if } \alpha > \frac{s}{s+1} \quad \text{(I)} \\ (1-\alpha T(s+1,s+1))^2 - (1-T(n,s))^2 & \text{if } \frac{s}{s+1} \geq \alpha > \frac{s-1}{s} \quad \text{(II)} \\ (1-\alpha T(s+1,s+1))^2 & \text{if } \frac{s-1}{s} \geq \alpha \quad \text{(III)} \end{cases}\end{aligned}$$

Denote the three possibilities here as $\Delta_k(n,s)$, $k = I, II, III$. We need to show that

$$\Omega(n,s) \equiv \Delta(n,s+1) - \Delta(n,s) < 0.$$

From Fig. 1, it is clear that several cases can occur when s increases.

- Case (I,I): $\Omega(n,s) = \Delta_I(n,s+1) - \Delta_I(n,s)$
- Case (I,II): $\Omega(n,s) = \Delta_{II}(n,s+1) - \Delta_I(n,s)$
- Case (II,III): $\Omega(n,s) = \Delta_{III}(n,s+1) - \Delta_{II}(n,s)$
- Case (III,III): $\Omega(n,s) = \Delta_{III}(n,s+1) - \Delta_{III}(n,s)$

The cases describe whether the increase in s stays in the same region (cases (I,I) and (III,III)), or transitions from I to II in case (I,II) or II to III in case (II,III). Region II is only valid for a single value of s, so is always a transitional region. We can write

$$\Omega(n,s) = \begin{cases} \Delta_I(n,s+1) - \Delta_I(n,s) & \text{if } \alpha > \frac{s+1}{s+2} \quad \text{(I,I)} \\ \Delta_{II}(n,s+1) - \Delta_I(n,s) & \text{if } \frac{s+1}{s+2} \geq \alpha > \frac{s}{s+1} \quad \text{(I,II)} \\ \Delta_{III}(n,s+1) - \Delta_{II}(n,s) & \text{if } \frac{s}{s+1} \geq \alpha > \frac{s-1}{s} \quad \text{(II,III)} \\ \Delta_{III}(n,s+1) - \Delta_{III}(n,s) & \text{if } \frac{s-1}{s} \geq \alpha > 0 \quad \text{(III,III)} \end{cases} \tag{A.2}$$

When $s = 0$, only the first two lines apply since region I is always the area from which the comparison begins. For $s = 1$, the third line must also be used, and the fourth is valid when $s > 1$.

Scenario (A): s=0.

Consider first $s = 0$, in which case we need to compare $\Delta(n,1)$ and $\Delta(n,0)$ where the latter is always given by $\Delta_I(n,0)$, and $\Delta(n,1) = \Delta_I(n,1)$ for $\alpha > \frac{1}{2}$ and $\Delta(n,1) = \Delta_{II}(n,1)$ for $\frac{1}{2} \geq \alpha$.

(A).(i). Suppose that $\alpha > \frac{1}{2}$ so that $\Omega(n,0) = \Delta_I(n,1) - \Delta_I(n,0)$.

Direct computation gives

$$\begin{aligned}\Delta_I(n,s{+}1) - \Delta_I(n,s) = {} & \frac{\alpha(1{-}\alpha)(n{-}1)}{[n-(1{-}\alpha)(s{+}1)][n-(1{-}\alpha)(s{+}2)]} \\ & \times \left\{\alpha(n{-}1)\cdot\frac{2n-(1{-}\alpha)(2s{+}3)}{[n-(1{-}\alpha)(s{+}1)][n-(1{-}\alpha)(s{+}2)]} - 2\right\} \\ & + \frac{(1{-}\alpha)(n{-}1)}{[n-(1{-}\alpha)s][n-(1{-}\alpha)(s{+}1)]} \\ & \times \left\{(n{-}1)\cdot\frac{2n-(1{-}\alpha)(2s{+}1)}{[n-(1{-}\alpha)s][n-(1{-}\alpha)(s{+}1)]} - 2\right\}. \end{aligned} \tag{A.3}$$

Inserting $s = 0$ into (A.3)

$$\Delta_I(n,1) - \Delta_I(n,0) = A + B,$$

where:

$$A = \frac{\alpha(1-\alpha)(n-1)}{(n-(1-\alpha))(n-2(1-\alpha))} \left[\alpha(n-1)\cdot\frac{2n-3(1-\alpha)}{(n-(1-\alpha))(n-2(1-\alpha))} - 2\right],$$

$$B = \frac{(1-\alpha)(n-1)}{n(n-(1-\alpha))}\left[(n-1)\cdot\frac{2n-(1-\alpha)}{n(n-(1-\alpha))} - 2\right].$$

Let us denote the inner bracket in A by

$$E := \alpha(n-1)\cdot\frac{2n-3(1-\alpha)}{(n-(1-\alpha))(n-2(1-\alpha))}.$$

We will show that $E < 2$, so that the entire bracket $[E-2]$ is negative.

Since $2n - 3(1-\alpha) < 2n$ for all $\alpha \in (0,1)$, we get the bound

$$E < \alpha(n-1)\cdot\frac{2n}{(n-(1-\alpha))(n-2(1-\alpha))}.$$

Moreover, using that $n-(1-\alpha) > \alpha$ and $n-2(1-\alpha) \geq 2\alpha$, we have:

$$E \leq \frac{2\alpha(n-1)}{(n-(1-\alpha))^2} < 2.$$

Thus, $E - 2 < 0$, and so $A < 0$.

Let us denote the inner bracket in B by

$$F := (n-1) \cdot \frac{2n-(1-\alpha)}{n\,(n-(1-\alpha))}.$$

Again, we show $F < 2$, so that $F - 2 < 0$.

Since $2n - (1-\alpha) < 2n$, we have

$$F < (n-1) \cdot \frac{2n}{n(n-(1-\alpha))} = \frac{2(n-1)}{n-(1-\alpha)}.$$

Now,

$$\frac{2(n-1)}{n-(1-\alpha)} < 2 \quad \Longleftrightarrow \quad 2(n-1) < 2(n-(1-\alpha)) = 2n - 2(1-\alpha),$$

which holds for all $0 < \alpha < 1$.

Hence, $F < 2$, so $F - 2 < 0$, and thus $B < 0$. Therefore, $A < 0$ and $B < 0$, and we conclude:

$$\Delta_I(n,1) - \Delta_I(n,0) = A + B < 0.$$

(A).(ii). Suppose now that $\frac{1}{2} \geq \alpha$ so that $\Omega\,(n,0) = \Delta_{II}\,(n,1) - \Delta_I\,(n,0)$.

Calculation gives

$$\begin{aligned}
&\Delta_{II}(n,s+1) - \Delta_I(n,s) \\
&\quad = \frac{1}{(2+s)^2} + 2(\alpha+1) \cdot \frac{n-1}{n-(1-\alpha)(s+1)} \\
&\quad - (\alpha^2+1) \cdot \left(\frac{n-1}{n-(1-\alpha)(s+1)}\right)^2 - 2 \cdot \frac{n-1}{n-(1-\alpha)s} + \left(\frac{n-1}{n-(1-\alpha)s}\right)^2.
\end{aligned}$$

Inserting $s = 0$,

$$\Delta_{II}(n,1) - \Delta_I(n,0) := D(n,\alpha) = \frac{1}{4} + \frac{1}{n^2} - \frac{\alpha^2 + [(1-\alpha)n - 1 + 2\alpha]^2}{(n-1+\alpha)^2}. \quad \text{(A.4)}$$

Differentiating w.r.t. α gives

$$\frac{\partial D}{\partial \alpha} = \frac{-2(n-1)}{(n-1+\alpha)^3} \underbrace{\left[\alpha n^2 - 3\alpha n + 3\alpha - n^2 + 2n - 1\right]}_{=:M(n,\alpha)}.$$

Write

$$M(n,\alpha) = -(1-\alpha)n^2 - (3\alpha - 2)n + (3\alpha - 1).$$

For $0 < \alpha \leq \frac{1}{2}$ the coefficients obey $1-\alpha \geq \frac{1}{2},\ 3\alpha - 2 \leq -\frac{1}{2},\ 3\alpha - 1 \leq \frac{1}{2}$. Hence, for every $n \geq 2$, $M(n,\alpha) \leq -\frac{1}{2}n^2 + n - \frac{1}{2} < 0$. Because $-2(n-1)/(n-1+\alpha)^3 < 0$,

we conclude $\frac{\partial D}{\partial \alpha} > 0$. This means that if $D(n, \frac{1}{2}) < 0$, then it will also be negative for $\frac{1}{2} > \alpha > 0$. Evaluating and simplifying gives:

$$D\big(n, \tfrac{1}{2}\big) = -\frac{1}{4} \cdot \frac{16n - 13n^2 + 4n^3 - 4}{n^2(2n-1)^2}.$$

The sign of $D\big(n, \frac{1}{2}\big)$ clearly depends on the numerator. For $n \geq 2$ the cubic $4n^3 - 13n^2 + 16n - 4$ is strictly positive (its first derivative is $12n^2 - 26n + 16 > 0$ and the value at $n = 2$ is 8). Hence

$$D\big(n, \tfrac{1}{2}\big) < 0 \quad \text{for all } n \geq 2.$$

By the argument above, $D(n, \alpha) < 0$ for $1/2 \geq \alpha > 0$.
Taking (A)(i) and (A)(ii) together shows that $\Omega(n, 0) < 0$ for all values of α.

Scenario (B): Larger s

We need to go through all four cases in (A.2).

Case (III, III)

This case is valid for $s \geq 2$. Here, $\Omega(n, s) = \frac{1}{(s+2)^2} - \frac{1}{(s+1)^2} < 0$. Hence, for $\frac{s-1}{s} \geq \alpha > 0$, $\Omega(n, s) < 0$ as required.

Case (II, III)

Direct calculation gives

$$\Omega(n, s) = \underbrace{\frac{1}{(s+2)^2} - \frac{1}{(s+1)^2}}_{X(s)<0} + \underbrace{\left(\frac{1-(1-\alpha)s}{n-(1-\alpha)s}\right)^2}_{Y>0}.$$

Because $\alpha \leq s/(s+1)$, $1 - (1-\alpha)s = \alpha s - (s-1) \leq 1/(s+1)$. Since $n \geq s+2$, $n - (1-\alpha)s \geq n - 1 \geq s + 1$. Hence

$$0 < Y \leq \frac{1}{(s+1)^4}.$$

Furthermore,

$$|X(s)| = \frac{2s+3}{(s+2)^2(s+1)^2}.$$

Then we can write,

$$\frac{Y}{|X(s)|} \leq \frac{(s+2)^2}{(2s+3)(s+1)^2} < 1 \qquad (s \geq 1),$$

since $(s+2)^2 < (2s+3)(s+1)^2$ for all $s \geq 1$.

Because $Y < |X|$, we have $\Omega(n, s) = X + Y < 0$.

Case (I, I)

In (A.3), let us write

$$\Omega(n, s) = C_1 P + C_2 Q$$

where

$$C_1 = \frac{\alpha(1-\alpha)(n-1)}{[n-(1-\alpha)(s+1)][n-(1-\alpha)(s+2)]} > 0,$$

$$C_2 = \frac{(1-\alpha)(n-1)}{[n-(1-\alpha)s][n-(1-\alpha)(s+1)]} > 0,$$

$$P = \alpha(n-1)\,\frac{2n-(1-\alpha)(2s+3)}{[n-(1-\alpha)(s+1)][n-(1-\alpha)(s+2)]} - 2,$$

$$Q = (n-1)\,\frac{2n-(1-\alpha)(2s+1)}{[n-(1-\alpha)s][n-(1-\alpha)(s+1)]} - 2.$$

We show that $P < 0$, $Q < 0$.
Let $\beta = 1-\alpha$ $(0 < \beta \le 1/(s+2))$ and $d_j = n - \beta(s+j)$ $(j = 1, 2)$. With

$$G(\beta) = (1-\beta)(n-1)\big[2n - \beta(2s+3)\big] - 2d_1d_2$$

we have $P(\beta) = G(\beta)/d_1d_2$.

Expanding $G(\beta)$ gives the convex quadratic $G(\beta) = a\beta^2 + b\beta - 2n$, where

$$a = (2s+3)n - 2(s+1)(s+2) - (2s+3) > 0, \quad b = n(2s+5) - 2n^2 + 2s + 3.$$

Evaluating at both endpoints gives $G(0) = -2n < 0$, and $\beta_{\max} = 1/(s+2)$:

$$G(\beta_{\max}) = -\frac{2(n-s-2)}{(s+2)^2} < 0 \ (n \ge s+2).$$

Thus $G(\beta) < 0$ for all admissible β, and with $d_1d_2 > 0$ we have $P = G/d_1d_2 < 0$.

Let us now check $Q < 0$. With $\beta := 1-\alpha$ and $d_0 = n - \beta s$, $d_1 = n - \beta(s+1)$, set

$$K(\beta) = (n-1)\big[2n - \beta(2s+1)\big] - 2d_0d_1, \qquad Q(\beta) = \frac{K(\beta)}{d_0d_1}.$$

Expanding gives the concave quadratic

$$K(\beta) = -2s(s+1)\beta^2 + (2s+1)(n+1)\beta - 2n.$$

Its vertex $\beta_* = \dfrac{(2s+1)(n+1)}{4s(s+1)}$ satisfies $\beta_* \ge \dfrac{1}{s+2}$, i.e. lies to the right of the region I interval $\big(0, 1/(s+2)\big]$. Hence the maximum of K on that interval occurs at $\beta_{\max} = 1/(s+2)$, where

$$K(\beta_{\max}) = -\frac{2\big(n-s-2\big)}{(s+2)^2} < 0 \quad (n \ge s+2).$$

Concavity then implies $K(\beta) < 0$ for all admissible β, and because $d_0 d_1 > 0$ we have $Q(\beta) < 0$ throughout region I.

With $C_1 > 0, C_2 > 0, P < 0, Q < 0$, we can conclude that

$$\Omega(n, s) = C_1 P + C_2 Q < 0.$$

Case (I, II)

Let $\beta := 1 - \alpha \in [\, 1/(s+2),\ 1/(s+1))$ and define

$$x = T(n, s+1) = \frac{n-1}{n - \beta(s+1)}, \qquad y = T(n, s) = \frac{n-1}{n - \beta s}.$$

The exact expression is

$$Z := \Delta_{II}(n, s+1) - \Delta_I(n, s) = \frac{1}{(s+3)^2} + (1-y)^2 - (1-x)^2 - (1-\alpha x)^2.$$

We first bound the expression:

$$(1-y)^2 - (1-x)^2 = (x-y)[2 - (x+y)].$$

We have

$$x - y = (n-1)\left(\frac{1}{n - \beta(s+1)} - \frac{1}{n - \beta s}\right) = \frac{(n-1)\beta}{[n - \beta(s+1)][n - \beta s]} \le \frac{\beta}{n-1},$$

because $n - \beta(s+j) \ge n - \beta(s+1) \ge n - 1$ for $j = 0, 1$ (using $\beta \le 1/(s+1)$ and $n \ge s + 2$).

Next, for the second factor, apply the harmonic-mean bound:

$$\begin{aligned} x + y &= (n-1)\left(\frac{1}{n - \beta(s+1)} + \frac{1}{n - \beta s}\right) \\ &\ge \frac{2(n-1)}{n - \beta(s+1) + n - \beta s} = \frac{2(n-1)}{2n - \beta(2s+1)}. \end{aligned}$$

Now,

$$2 - (x+y) \le 2 - \frac{n-1}{n - \beta(s+1)} = \frac{1 - \beta(s+1)}{n - \beta(s+1)} \le \frac{1}{s+1},$$

since $n \ge s+2$ and $\beta \ge 1/(s+2)$ imply $n - \beta(s+1) \ge s+1$, and $1 - \beta(s+1) \le 1$.

Therefore,

$$(x-y)[2 - (x+y)] \le \frac{\beta}{n-1} \cdot \frac{1}{s+1} \le \frac{1}{(s+1)(s+2)}.$$

Because $\alpha \leq \dfrac{s+1}{s+2}$ and $x < 1$, we have

$$1 - \alpha x \geq \frac{1}{s+2} \quad \Rightarrow \quad (1-\alpha x)^2 \geq \frac{1}{(s+2)^2}.$$

Combining all bounds:

$$Z < \frac{1}{(s+3)^2} + \frac{1}{(s+1)(s+2)} - \frac{1}{(s+2)^2} = -\frac{2s+4}{(s+3)^2(s+2)^2} < 0.$$

Therefore $\Delta_{II}(n, s+1) - \Delta_I(n, s) < 0$ whenever $n \geq s+2$, $s \geq 1$ and $s/(s+1) < \alpha \leq (s+1)/(s+2)$. □

Proof of Proposition 3 We have for each region:

$$\Delta_I = (1 - \alpha\, T(n, s+1))^2 - (1 - T(n, s))^2, \tag{A.5}$$

$$\Delta_{II} = \frac{1}{(1+s)^2} - (1 - T(n, s))^2, \tag{A.6}$$

$$\Delta_{III} = \frac{1}{(1+s)^2}. \tag{A.7}$$

From (A.7) we see that the expression is independent of α, proving part (ii) of the proposition.

Let $T_\alpha(n, s)$ be the derivative of $T(n, s)$ with respect to α:

$$T_\alpha(n, s) = -\frac{(n-1)^2}{(n - (1-\alpha)s)^2} < 0.$$

Differentiating (A.5) and (A.6) with respect to α gives

$$\frac{\partial \Delta_I}{\partial \alpha} = -2\big(1 - \alpha\, T(n, s+1)\big)\left[\frac{(n-1)(n-(s+1))}{(n-(1-\alpha)(s+1))^2}\right] + 2(1 - T(n, s))T_\alpha(n, s) < 0, \tag{A.8}$$

$$\frac{\partial \Delta_{II}}{\partial \alpha} = 2(1 - T(n, s))T_\alpha(n, s) < 0. \tag{A.9}$$

The sign in (A.9) follows from the fact that $T_\alpha(n, s) < 0$; this sign in (A.8) follows from the same condition and noticing that the first three elements are all positive, preceded by a negative sign.

Furthermore,

$$\frac{\partial \Delta_I}{\partial \alpha} - \frac{\partial \Delta_{II}}{\partial \alpha} = -2\big(1 - \alpha\, T(n, s+1)\big)\left[\frac{(n-1)(n-(s+1))}{(n-(1-\alpha)(s+1))^2}\right] \leq 0 \tag{A.10}$$

where the equality holds if $n = s + 1$. □

Proof of Proposition 5 Fix an integer $n \geq 2$. The following observations will be useful in proving the proposition.

Observation 1 Suppose x is a discrete random variable with a probability mass function $p(x)$. Suppose $f(x)$ is a decreasing function of x and $g(x)$ is an increasing function of x with $\mathbb{E}_x[g(x)] = 0$. Then, $\mathbb{E}_x[f(x) \cdot g(x)] \leq 0$.

Proof of Observation 1 Define $X^+ := \{x : g(x) > 0\}$ and $X^- := \{x : g(x) \leq 0\}$. Since $\mathbb{E}_x[g(x)] = 0$, it follows that

$$\sum_{x \in X^+} g(x)p(x) = \sum_{x \in X^-} (-g(x))p(x).$$

Because g is increasing, every $x \in X^-$ is less than every $x \in X^+$. Moreover, since f is decreasing, it then follows that every $f(x), x \in X^+$ is less than every $f(x), x \in X^-$. Therefore,

$$\begin{aligned} &\sum_{x \in X^+} f(x)g(x)p(x) \leq \sum_{x \in X^-} f(x)(-g(x))p(x) \\ &\quad \Leftrightarrow \mathbb{E}_x[f(x)g(x)] \leq 0. \end{aligned}$$

This completes the proof of Observation 1. □

Observation 2 Suppose $s \backsim$ Binomial $(n-1, t)$ and $f(t, s)$ is a continuous function of $s \geq 0$ and $t \in (0, 1)$. Then,

$$\frac{d}{dt}\mathbb{E}_s[f(t,s)] = \mathbb{E}_s\left[f(t,s)\frac{s-(n-1)t}{t(1-t)} + \frac{d}{dt}f(t,s)\right].$$

Proof of Observation 2 Applying

$$\frac{d}{dt}t^s(1-t)^{n-s-1} = t^s(1-t)^{n-s-1}\left[\frac{s-(n-1)t}{t(1-t)}\right],$$

we get

$$\begin{aligned} \frac{d}{dt}\mathbb{E}_s[f(t,s)] &= \sum_{s=0}^{n-1}\binom{n-1}{s}t^s(1-t)^{n-s-1}\left[f(t,s)\cdot\frac{s-(n-1)t}{t(1-t)} + \frac{d}{dt}f(t,s)\right] \\ &= \mathbb{E}_s\left[f(t,s)\frac{s-(n-1)t}{t(1-t)} + \frac{d}{dt}f(t,s)\right]. \end{aligned}$$

This completes the proof of Observation 2. □

By Lemma 1 we know $\Delta(n, s) > 0$, strictly decreasing in s. For any candidate threshold $\theta \in (0, 1)$ write

$$H(t) = \sum_{s=0}^{n-1} \binom{n-1}{s} t^s (1-t)^{n-1-s} \Delta(n, s), \qquad t := q\theta. \tag{A.11}$$

The *interior* threshold $\theta_{\text{int}}(q)$ is determined by the fixed–point condition

$$q\, H\big(q\theta_{\text{int}}(q)\big) = \frac{\theta_{\text{int}}(q)}{k}. \tag{A.12}$$

Because $H(q\theta_{\text{int}}(q))$ is defined only if $q\,\theta_I \in (0, 1)$, the condition (A.12) yields an admissible solution only when $\theta_{\text{int}}(q) < \frac{1}{q}$. However, since $\frac{1}{q} > 1$, any admissible $\theta_{\text{int}}(q)$ coincides with the investment threshold θ_I whenever it below 1. Implicit differentiation of (A.12) yields, for any admissible solution $\theta_{\text{int}}(q)$,

$$\theta'_{\text{int}}(q) = \frac{H(t) + t\, H'(t)}{1/k - q^2 H'(t)}, \qquad t = q\theta_{\text{int}}(q).$$

Claim 1 For every $t \in (0, 1)$, $H(t)$ is decreasing.

Proof of Claim 1 We apply Observation 2 with $f(t, s) = \Delta(n, s)$. Then, $\frac{d}{dt} f(t, s) = 0$. Therefore,

$$H'(t) = \frac{1}{t(1-t)} \mathbb{E}_s\Big[\Delta(n, s)(s - (n-1)t)\Big].$$

Because $\Delta(n, s)$ is decreasing in s while $(s - (n-1)t)$ is increasing and $\mathbb{E}(s - (n-1)t) = 0$, it follows from Observation 1 that $H'(t) \leq 0$. This completes the proof of Claim 1.

Define

$$N(t) := H(t) + t\, H'(t), \qquad t \in (0, 1).$$

By Claim 1, $H'(t) \leq 0$, and therefore,

$$sign(\theta'_{\text{int}}(q)) = sign(N(t)).$$

The following result shows that $N(t)$ can change its sign at most once in the interval $(0, 1)$.

Claim 2 $N(t)$ has at most one zero in the interval $(0, 1)$.

Proof of Claim 2 Using the expression of $H'(t)$ derived in Claim 1, we write

$$N(t) = H(t) + tH'(t) = \frac{1}{1-t} \mathbb{E}_s[\Delta(n, s)(s + 1 - nt)]. \tag{A.13}$$

Observe that $\mathbb{E}_s[s+1-nt] = (1-t)$. For given $t \in (0,1)$, define $\hat{S}(t)$ is the largest $s \in \{0, 1, \ldots, n-1\}$ such that the conditional expectation of $(s+1-nt)$ subject to $s \leq \hat{S}(t)]$ is negative, i.e., $\mathbb{E}_s[s+1-nt | s \leq \hat{S}(t)] \leq 0$. Since $s+1-nt$ is increasing in s and decreasing in t, $\hat{S}(t)$ is weakly increasing in t. It also follows that $\hat{S}(t) > nt-1$, since for all $s \leq nt-1$, $s+1-nt \leq 0$, and $\mathbb{E}_s[s+1-nt | s \leq nt-1] < 0$.

Now, we can express

$$\mathbb{E}_s[\Delta(n,s)(s+1-nt)] = \underbrace{\mathbb{E}_s[\Delta(n,s)(s+1-nt) \mid s \leq \hat{S}(t)]}_{A} \Pr[s \leq \hat{S}(t)] + \underbrace{\mathbb{E}_s[\Delta(n,s)(s+1-nt) \mid s > \hat{S}(t)]}_{B} \Pr[s > \hat{S}(t)]$$

Since $\mathbb{E}_s[s+1-nt | s \leq \hat{S}(t)] \leq 0$, by Observation 1, $A \leq 0$. Moreover, since $\hat{S}(t) > nt-1$, we have $B > 0$. Both A and B are decreasing in t, since $(s+1-nt)$ decreases with t and $\hat{S}(t)$ is increasing in t.

Further, as t increases, $\Pr[s \leq \hat{S}(t)]$ rises and $\Pr[s > \hat{S}(t)]$ falls, so the overall term—$\mathbb{E}_s[\Delta(n,s)(s+1-nt)]$—becomes progressively less negative. Therefore, once the sign of $N(t)$ becomes negative, it remains so for higher values of t. This completes the proof of Claim 2.

To determine $sign(N(t))$, observe that $\lim_{t \to 0^+} N(t) = \Delta(n,0) > 0$. It therefore follows from Claim 2 that either $N(t) > 0$ for all $t \in (0,1)$, or there exists a threshold $t^* \in (0,1)$ such that $N(t) > 0$ for all $t < t^*$ and $N(t) < 0$ for all $t > t^*$.

Because $sign(\theta'_{\text{int}}(q)) = sign(N(t))$, $\theta'_{int}(q)$ changes its sign at most once—from positive to negative—implying that $\theta_{int}(q)$ is unimodal. If $N(t) > 0$ for all t, then $\theta_{int}(q)$ is strictly increasing on $(0,1)$; otherwise, $\theta_{int}(q)$ is increasing in q up to a threshold q^*, and is decreasing thereafter. □

Proof of Proposition 6 First, we show that $q\theta_I$ is strictly increasing in q. Let $p = q\theta_I$. When $\theta_I = 1$, it trivially follows that p is strictly increasing in q.

When $\theta_I < 1$, by Proposition 1, it satisfies

$$q\mathbb{E}_s[\Delta(n,s)] - \frac{\theta_I}{k} = 0,$$

where s follows Binomial$(n-1, p)$. Differentiating the above equation with respect to q, we get

$$\mathbb{E}_s[\Delta(n,s)] + q\frac{d\mathbb{E}_s[\Delta(n,s)]}{dp}\frac{dp}{dq} - \frac{1}{k}\frac{d\theta_I}{dq} = 0.$$

Replacing $\frac{d\theta_I}{dq}$ by $\frac{1}{q}[\frac{dp}{dq} - \theta_I]$ and after arranging terms, we have

$$\frac{dp}{dq} = \frac{\mathbb{E}_s[\Delta(n,s)] + \frac{\theta_I}{qk}}{\frac{1}{qk} - q\frac{d\mathbb{E}_s[\Delta(n,s)]}{dp}}.$$

Since $\Delta(n,s)$ is decreasing in s, by Fact 1, $\frac{d\mathbb{E}_s[\Delta(n,s)]}{dp} < 0$, which implies that $\frac{dp}{dq} > 0$.

Since by Lemma 3, T is strictly increasing in the number of skilled player s, which is distributed as Binomial(n, p), we further apply Fact 1 to conclude that $\mathbb{E}_s[T]$ is strictly increasing in p. Therefore, $\frac{d\mathbb{E}_s[T]}{dq} = \frac{d\mathbb{E}_s[T]}{dp}\frac{dp}{dq} > 0$. □

Proof of Proposition 7 Note that the investment threshold is the minimum of the implicit solution of (20) and 1. Below we show that the implicit solution of (20) is weakly decreasing in m, which also implies that the threshold is weakly decreasing in m. Define

$$\begin{aligned} F(m) &:= \sum_{s=0}^{n-m-1} \binom{n-m-1}{s} (q\theta_I)^s (1-q\theta_I)^{n-m-1-s} \Delta(n, s+m) \\ &= \mathbb{E}_{X_m}(\Delta(n, X_m + m)), \end{aligned}$$

where X_m denotes a variable following the Binomial distribution with $n-1-m$ trials and a success probability of $q\theta_I$. Observe that $\mathbb{E}_{X_{m-1}}(X_{m-1}) = q\theta_I \mathbb{E}_{X_m}(X_m + 1) + (1 - q\theta_I)\mathbb{E}_{X_m}(X_m)$, and therefore

$$\begin{aligned} F(m-1) &= \mathbb{E}_{X_{m-1}}(\Delta(n, X_{m-1} + m - 1)) \\ &= q\theta_I \mathbb{E}_{X_m}(\Delta(n, X_m + m)) + (1 - q\theta_I)\mathbb{E}_{X_m}(\Delta(n, X_m + m - 1)) \\ &= q\theta_I F(m) + (1 - q\theta_I)\mathbb{E}_{X_m}(\Delta(n, X_m + m - 1)) \end{aligned}$$

Because $\Delta(n, s+k)$ is decreasing in s for any fixed value of $k \in \{0, 1, \ldots, n-1-s\}$, it follows from Fact 1 that $\mathbb{E}_s(\Delta(n, s+m-1))$ is decreasing in the number of trials of the Binomial variable s.

Therefore, $\mathbb{E}_{X_{m-1}}(\Delta(n, X_{m-1} + m - 1)) \leq \mathbb{E}_{X_m}(\Delta(n, X_m + m - 1))$.

Hence, $F(m-1)$ is a weighted average of $F(m)$ and another number larger than $F(m-1)$, implying that $F(m-1)$ must be larger than $F(m)$. Further, it follows from (20) that $\theta_I(m)$ is an implicit solution of the equation $kqF(m) - \theta_I = 0$. Since $F(m, \theta) \leq F(m-1, \theta)$ for any θ, it implies that $\theta_I(m) \leq \theta_I(m-1)$. □

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