Journal of Mathematical Economics 122 (2026) 103203

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Endogenous entry and investment in an all-pay auction

Derek J. Clark [a], Tapas Kundu [b,*]

[a] School of Business and Economics, UiT - The Arctic University of Norway, Norway
[b] Oslo Business School, Oslo Metropolitan University, Norway

ARTICLE INFO

Manuscript handled by Editor C. Carmen Bevia

JEL classification:
D02
D72
D81
D82

Keywords:
All-pay contest
Investment
Endogenous entry
Threshold strategy

ABSTRACT

Contests are ubiquitous but do not simply arise in a vacuum. Competitors make conscious decisions before the fighting stage. This paper looks at the interplay between the decision to enter and then undertake a pre-contest investment to enhance the chance of winning the prize. We use an all-pay auction to model the contest stage; investment cost is private information and its return is stochastic. We characterize equilibrium in terms of threshold strategies on the cost parameter, both for the entry and the subsequent investment decision. We show that the stochastic nature of the investment outcome has a non-monotonic effect on players decisions in equilibrium. A contest designer can use our results to directly achieve goals related to the maximization of contest effort, entry fee revenue and investment propensity. In all cases, we demonstrate that limiting entry may be optimal.

1. Introduction

A contest captures strategic interaction between rivals for a prize in which the costs of actions to win are sunk. Examples come from many areas such as political science (e.g. lobbying, armed conflict), economics (e.g. promotions or advertising campaigns) and sports.[1] Much literature assumes that a contest arises among a set of defined competitors with certain characteristics. This paper examines the interplay between factors that may affect actions in the contest prior to the fighting stage. First, we open for the possibility that the set of participants is not known at the beginning of the competition. Second, entrants may be able to undertake a pre-contest action that potentially improves the probability of winning and/or the size of the prize. Third, the return to any such action may be uncertain.

We develop a tractable three-period model, involving entry, pre-contest investment and then the actual contest. At the beginning, a fixed set of players are privately informed of their marginal cost of making the pre-contest investment, and must decide whether or not to enter the competition. Next, upon entry, they decide whether to invest in acquiring an advantage which would create a favorable imbalance in the future contest. The return to investment in advantage acquisition is uncertain. Finally, the set of entrants, some of which have potentially acquired an advantage over their rivals, compete in an all-pay auction to win the prize.

Our findings demonstrate the complex interplay between entry and investment. In making the entry decision, an agent is enticed by the potential value of the contest prize, and upon entry weighs up the private cost of investment with its return. Even if an investment is successful, actions at the contest stage may dissipate its return. The exception is if an agent is the only one to succeed in its investment. Intuitively, contests are hard fought between a sufficiently homogeneous group of competitors, but a single strong player can dissuade rivals from making effort. This forms the incentive to enter and to invest, and we show how this is critically determined by the relationship between the number of agents and the probability of successful investment. Entry and investment are least attractive when the probability of investment success is very high or low, since this imparts a low expectation of being the single strong player (with a successful investment) at the contest stage. Agents are more likely to enter and invest if the success probability takes an intermediate value.

We show also that the entry cost acts as a mechanism to exclude less efficient agents from entering the contest. A designer can effectively use the entry fee to exclude high-cost agents, even though cost information is not freely available.[2] Furthermore, we show how the entry

* Corresponding author.
E-mail addresses: derek.clark@uit.no (D.J. Clark), tapas.kundu@oslomet.no (T. Kundu).

[1] See Konrad (2009) for an overview of contest types and applications.

[2] In a similar vein, Fu et al. (2015) show that an effort-maximizing contest designer may wish to limit participation in a Tullock contest with homogeneous participants with endogenous (and concealed), costly entry.

https://doi.org/10.1016/j.jmateco.2025.103203
Received 8 May 2025; Received in revised form 11 November 2025; Accepted 12 November 2025

cost may be set in order to achieve a vested interest that the designer may have such us maximizing expected contest effort, entry revenue or the expected number of investing agents.

Our analysis can be motivated by application and by gaps in the literature. A relevant application is competition to become a technological standard or a leading technology which can exhibit the characteristics captured by our stylized model: winner-take-all prizes, an unknown set of participants and actions that are taken to give a competitive edge. Technology markets typically exhibit winner-take-all properties. Marc Andreessen, a Silicone Valley veteran and co-founder of Netscape has stated: "The big companies, though, in technology tend to have 90 percent market share. So we think that generally these are winner-take-all markets. Generally, number one is going to get like 90 percent of the profits. Number two is going to get like 10 percent of the profits, and numbers three through 10 are going to get nothing".[3] Technology markets are often built on a direct network externality, arising since users get more utility from using a service that connects many others. Besen and Farrell (1994) characterize network markets as "tippy", noting that small differences in products, platforms or technologies – whether real or merely perceived – "can be magnified in a process in which some firms make extremely large gains and in which dominant market positions are difficult to change" (p. 119). Technology advancement affects other markets, and the fight for dominance in the entertainment industry is ongoing. As The Economist noted in 2017: "The best time to gain (or lose) audience — and to challenge the dominance of an established platform — is when technology makes a leap. That is why media, gaming and tech companies are investing billions in virtual reality and augmented reality".[4] The next winner-take-all contest in the entertainment industry is, predicted to have a value of nearly 1 trillion USD by 2030, to the firm that conquers the metaverse, the next big platform incorporating "extended reality" and Artificial Intelligence (AI).[5]

A novel technology often opens new market opportunities, taking time before the set of participants is established. In the fledgling market for home video entertainment at the beginning of the 1970's several companies entered, each with their own technology. Sony launched its Betamax videotape recording system, the Victor Company of Japan (JVC) pioneered the Video Home System (VHS) format, while RCA and Philips developed needle-based record-style discs.[6] It is now well known that the main war was between Betamax and VHS, with the latter claiming the market standard. The companies sunk large resources into making hardware improvements and convincing film studios and distributors to choose their format. The outcome of these investments was uncertain; Betamax was widely regarded as a superior system in terms of picture and sound quality, but the availability of longer recording times on a VHS cassette seems to have tipped the balance, deciding the outcome of the contest in favor of that format.[7]

Uncertainty over the number and quality of rivals is present in new markets such as cryptocurrency mining, which involves introducing new blocks of a currency to its existing circulation using a consensus mechanism.[8] Prior to fighting for the final prize, technology rivals can undertake various types of investment in order to improve their ultimate chance of victory. The returns to the investment are often uncertain. Cryptocurrency miners compete against each other to solve a cryptographic puzzle, with the winner receiving the block reward. The cost and efficiency of mining equipment can vary significantly. For example, a general-purpose central processing unit (CPU) may not be expensive, but it is inefficient in solving the puzzle (Narayanan et al., 2016). A network of CPUs has potential to reduce the search time, but the associated electricity costs make it an impractical pursuit. Professional miners therefore invest in more efficient but expensive networks of application-specific integrated circuits (ASICs), which can only mine one type of cryptocurrency. Miners are confronted with uncertainty regarding the return on their investment in hardware equipment (Halaburda et al., 2022). At the time of investment, they are uncertain about the future market value of the specific cryptocurrency that the ASIC can mine.

In addition to finding real-world applications, our analysis also unites disparate strands of the contest literature. The interaction of the entry and investment decisions appears to have been little studied previously. Investment may enhance the size of a prize that can be appropriated as in Konrad (2002),[9] or reduce the cost of competing for the prize (Fu and Lu, 2009 and Münster, 2007). Several contest papers consider the possibility that rivals make endogenous entry decisions as part of the equilibrium strategy (for example Fu and Lu, 2010; Fu et al., 2015; Liu and Lu, 2019; Jiao et al., 2022 and Kaplan and Sela, 2010).[10] Some work has been done on the disclosure or concealment of the set of entrants (Fu et al., 2011; Jiao et al., 2022). In an all-pay auction with different assumptions about entry, Chen et al. (2017) show that a revenue-maximizing principal prefers to conceal (disclose) the number of entrants if the cost of effort is convex (concave); with a resource constraint on the bid, Chen et al. (2020) find that a principal will always prefer to conceal information on entry.[11]

Another key feature of our analysis is the cost of entry. Kaplan and Sela (2010) and Liu and Lu (2019) study an all-pay auction with an entry cost. Ability is common knowledge and higher ability gives a reduced entry cost and effort cost in the former, while it is private information in the latter but does not affect the cost of entry. In common with our approach, these papers derive a threshold strategy for entry into the contest. Kaplan and Sela (2010) demonstrate that the contest may not be effective in the sense that the probability of entry is not increasing in ability. To rectify this, they show that the winner of the contest can be charged a fee, although this reduces the attractiveness of entry for all competitors. Liu and Lu (2019) focus attention on the division of the prize mass into different sizes, showing that a single winner-take-all prize is optimal if the cost of effort is linear or concave. The first stage of the model is similar to the contest entry game in Hammond et al. (2019), who analyze an all-pay auction between players with different costs of effort that are private information. Assuming that the entry fee augments the prize, they derive an elegant solution for the entry threshold and total effort. In addition to deriving a threshold for entry, our analysis introduces further uncertainty by considering which of the entrants that will make an ability-enhancing investment before the actual contest is played.

Pre-contest play has been studied to a modest extent, most commonly set in terms of rivals being able to take a single decision before fighting at the contest stage. In isolation, modeling the pre-contest investment return as stochastic complements the deterministic approach of for example Münster (2007). Our results are easily adapted to study a situation in which no conscious entry decision is made and no entry cost is incurred; the set of rivals is given at the outset. In other applications, an entry stage is a natural part of the market to be studied. One example is procurement contests for the US Department

[3] See Tsotsis (2013).

[4] See The Economist (2017).

[5] See Koohang et al. (2023).

[6] See Wikipedia (2023).

[7] A standard Betamax tape lasted 60 min, whereas a 3-hour VHS cassette was capable of recording a whole movie (Owen, 2005).

[8] See Narayanan et al. (2016) and Sriman et al. (2021) for consensus mechanisms.

[9] Konrad (2002) only considers pre-contest investment by the incumbent.

[10] These issues have parallels in the auction literature. Gentry et al. (2017) characterize equilibrium entry when bidders incur entry costs, exploring the trade-off between revenue maximization and efficiency. Chen et al. (2025) show how selective entry arises and can be identified in first-price auctions with risk-averse bidders.

[11] A further strand of the literature considers whether the investment decision is observable (taken simultaneously or sequentially as in Münster, 2007).

of Defense in which potential suppliers often have to participate in a training or support program in order to be able to bids on contracts.[12] Theoretically, our framework demonstrates the interaction between the entry and investment decisions, where entry reveals useful information about rivals which affects the overall return to investment. In innovation contests, a sponsor can experience that increasing the set of entrants will decrease effort per competitor, but this may also increase the breadth of the R&D approach, providing more diverse solutions and a higher probability of an extreme value solution (Terwiesch and Xu, 2008; Boudreau et al., 2011).[13]

The rest of the paper is organized as follows. The basic model framework is presented in Section 2, and Section 3 sets up the contest. Pre-contest investment decisions are analyzed in Section 4, while Section 5 considers entry strategy. The significance of the model parameters for the analysis is expounded in Section 6. Section 7 considers the achievement of different goals by a contest designer, and Section 8 concludes. All proofs are in Appendix A.

2. Model

Consider $N \geq 2$ agents who can enter an all-pay auction by incurring an entry cost $c > 0$. Denote the set of entrants by E, which has $n \leq N$ members. After entering, agents can invest in acquiring an advantage that affects their valuation of the contest prize. In this model, there are two equivalent ways of modeling how a successful investment affects payoffs; one can either assume that successful agents have a lower marginal cost of exerting effort in the contest, or that such an agent has a larger prize value.[14] We shall follow the latter interpretation, so that a successful agent has a value of winning the contest of αv where $\alpha > 1$ is common knowledge; the valuation of an unsuccessful or non-investing agent is v. Agent $i \in E$ has an investment cost of θ_i, where $\theta_i, i \in E$ are independent random variables uniformly distributed over $[0, 1]$.[15] The return to investment is stochastic. The likelihood of success is $q \in (0, 1)$, which is the same for every investor and is common knowledge. Those who do not invest do not acquire the advantage. We denote by $m \leq n$ the number of agents that realize a successful return.

The agents that enter the contest exert efforts, given by $\mathbf{x} = (x_1, x_2, \ldots .., x_n)$, in order to win the contest prize. In an all-pay auction, the winner is the contestant with the highest effort; if several agents have the same maximal effort, they each have an equal probability of winning. Let $W(\mathbf{x}) = \left\{ j \in E \mid x_j \geq x_z \text{ for every } z \in E \right\}$ represent the set of agents that have maximal effort. The probability of agent $i \in E$ winning the contest is given by

$$p_i(\mathbf{x}) = \begin{cases} \frac{1}{|W(\mathbf{x})|} & \text{if } i \in W(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases}$$

The expected payoff of agent $i \in E$ with prize $V \in \{v, \alpha v\}$ is

$$\pi_i = p_i(\mathbf{x})V - x_i. \tag{1}$$

The game proceeds follows:

Table 1
Key notation.

Symbol	Meaning
N	Potential agents; n entrants; m successful investors
θ	Individual investment cost; private type
$\theta_E,\ \theta_I$	Entry and investment cutoffs (thresholds)
q	Success probability of investment
$v,\ \alpha v$	Baseline and augmented prize
$\pi(\theta, n)$	Stage-2 continuation payoff from entering (Eqs. (7)–(10))
$\xi(n)$	Gross expected return under full investment (Eq. (6))

- Stage 0: Nature chooses the type θ_i of agent $i = 1, 2, \ldots, N$. The type of an agent is private information.
- Stage 1 (Entry): Agent $i \in \{1, 2, \ldots, N\}$ decides whether to enter, after paying the entry fee. The subset of entering agents is E with $|E| = n$ as the number of entrants; n is public information. If $n = 0$, the game ends.
- Stage 2 (Investment): Agent $i \in E$ decides whether to invest by incurring an investment cost of θ_i. Then, nature decides whether an agent realizes a successful investment. The number of agents that realize a successful return, $m \leq n$, is public information.
- Stage 3 (Contest): If $n = 1$, the sole entrant wins the prize. If $n > 1$, agents participate in an all-pay auction to win the prize.

Agents have private information on their own investment cost at the start of the game, and we assume that the entry decision and investment outcome is public information. This is in line with the increasing popularity of "unblind" contests that allow rivals identities and actions to be known (Bockstedt et al., 2022).[16] We study the perfect Bayesian equilibrium of the game in symmetric strategies.[17]

Table 1 gives an overview of the main notation used in the paper.

3. Contest stage

We begin our analysis at stage 3, where the number of entrants n and the number of successful agents m are common knowledge. Denote the expected contest-stage payoff of a successful agent by $\pi_s(n, m)$, and an unsuccessful one by $\pi_u(n, m)$.[18] Let $T(n, m)$ be the total expected effort exerted in the contest.

If $n = 1$, there is no contest and the sole entrant wins the prize. The entrant's payoff is αv if its investment has paid off, and v if it did not invest, or if the investment failed. We can therefore set $\pi_s(1, 1) = \alpha v$, $\pi_u(1, 0) = v$, and $T(1, 1) = T(1, 0) = 0$.

Consider $n \geq 2$ and $m \in \{0, 1, \ldots, n\}$. Each agent chooses effort to maximize (1). This is then a standard all-pay auction under complete information, which has been extensively studied by Baye et al. (1996). We can use their results directly. There are three cases to consider[19]:

[12] See Office of the Under Secretary of Defense for Acquisition and Sustainment, US Department of Defense (2022).

[13] In line with our analysis, Stouras et al. (2022) state that uncertainty over the number of entrants is a primary feature of innovation contests, and consider when a principal should offer several rewards in order to attract a sufficient number of talented solvers.

[14] See Vojnović (2015). In an early paper, Konrad (2002) considered an investment made by a single agent – the incumbent – that increased the size of the prize that this agent and the single rival could fight over. Our model permits investment by all agents, and its return accrues as a private benefit.

[15] This simplifies the analysis, but the results generalize to any atomless type distribution. See the discussion at the end of Section 5.

[16] We consider how our main analysis changes with less public information at the end of Section 5.

[17] Asymmetric equilibria can arise in endogenous–entry contests, where some potential entrants may or may not enter regardless of circumstances, while others follow a threshold-based entry strategy. We analyze the symmetric equilibrium, as it serves as a natural focal point in the absence of a clear coordination mechanism, given that investment type is private information and all firms face identical entry costs. Symmetric equilibria have also been the primary focus of the theoretical literature on contests with endogenous entry; see Fu et al. (2015).

[18] At this stage, an agent may be "unsuccessful" because it did not invest or because it did invest but not succeed. Whatever the source of this lack of success, any investment cost is sunk and this does not affect payoffs at the contest stage.

[19] Cases (i) and (ii) use Theorem 1 in Baye et al. (1996), and case (iii) uses Theorem 2.

Lemma 1 (Baye et al., 1996). *Suppose that $n \geq 2$.*

(i) $m = 0$. Then $\pi_u(n,0) = 0$, $T(n,0) = v$.

(ii) $n \geq m \geq 2$. Then $\pi_s(n,m) = \pi_u(n,m) = 0$, $T(n,m) = \alpha v$.

(iii) $m = 1$. Then $\pi_s(n,1) = (\alpha-1)v$, $\pi_u(n,1) = 0$, and $T(n,1) \in \left[T^{min}(n,1), T^{max}(n,1)\right]$ where

$$T^{min}(n,1) = \frac{v}{n}\left[(n-1)^2\alpha^2 - (2n-1)(n-1)\alpha + n^2 - (n-1)^2(\alpha-1)^{\frac{2n-1}{n-1}}\alpha^{\frac{-1}{n-1}}\right], \tag{2}$$

$$T^{max}(n,1) = \frac{\alpha+1}{2\alpha}v. \tag{3}$$

In case (i), no agent has realized a successful investment, and all expect a payoff of zero since they exert an expected amount of effort in aggregate that equals the value of the prize. In case (ii), there are at least two successful agents and these exert efforts that are expected to equal their prize value αv. Agents that have not acquired the investment advantage do not exert effort, and all participants expect a payoff of zero. In both of these cases, there are a continuum of equilibria, but Baye et al. (1996) show that they all lead to the same expected total effort. A single successful agent – as in case (iii) – will have a positive expected payoff equal to the difference in the prize between it and a rival with that has not acquired the investment advantage. All unsuccessful agents expect a payoff of zero. Again, there are a continuum of equilibria also in this case, but they do not lead to the same amount of aggregate expected effort. $T(n,1)$ is minimized when the unsuccessful agents all compete in the contest, using a symmetric strategy; this leads to expected effort $T^{min}(n,1)$. On the other hand, Baye et al. (1996) show that $T(n,1)$ is maximized when all but one of the unsuccessful agents have an effort of zero, yielding an expected effort of $T^{max}(n,1)$. When $n = 2$, these equilibria coincide since both imply that one successful agent competes with one unsuccessful rival, and $T(2,1) = T^{min}(2,1) = T^{max}(2,1)$.

Because $\alpha > 1 > \frac{\alpha+1}{2\alpha}$, the principal always gets the lowest effort when there is exactly one successful agent, and highest when there is more than one. We state this result formally:

Corollary 1. $T(n, m \geq 2) > T(n,0) > T(n,1)$.

4. Investment

Consider stage 2, where the number of entrants n is common knowledge. The investment strategy affects the number of successful players. An agent's investment decision is contingent on his type θ. If there are $m-1 \in \{0,1,\ldots,n-1\}$ successful agents among the other $n-1$ rivals, the return to investment for an agent of type θ is $q\,\triangle(n,m) - \theta$, where

$$\triangle(n,m) = \pi_s(n,m) - \pi_u(n,m-1).$$

An agent's expected return to investment is $q\mathbb{E}_{m-1}(\triangle(n,m)) - \theta$, where the expectation is taken over the probability distribution of $(m-1)$, the number of successful players among $(n-1)$ competitors.

As investment success is a binary event in our model, the number of successful players follows a Binomial distribution. Specifically, $(m-1) \sim Binomial(n-1,\kappa)$ where κ is the probability of finding a successful agent conditional upon entry. This probability κ depends on the investment and entry strategies of the players.

Proposition 1 below shows that every agent's optimal investment strategy is a threshold strategy whenever all agents follow a threshold entry strategy. The intuition is straightforward – conditional upon entry, every agent's expected payoff is decreasing in its own investment cost, and therefore investment pays off only if the cost is sufficiently low. In this case, $\kappa = q\theta_I/\theta_E$, where $\theta_I \in [0,\theta_E]$ and $\theta_E \in [0,1]$ denote the investment and entry thresholds respectively.

Given θ_E, we can derive the investment threshold from the investment-indifference condition:

$$q\left[\sum_{m-1=0}^{n-1}\binom{n-1}{m-1}\left(\frac{q\theta_I}{\theta_E}\right)^{m-1}\left(1-\frac{q\theta_I}{\theta_E}\right)^{n-m}\triangle(n,m)\right] - \theta_I = 0. \tag{4}$$

Recall that the term $\Delta(n,m) = \pi_s(n,m) - \pi_u(n,m-1)$ is the extra contest-stage value from being the additional successful investor when $m-1$ of the other $n-1$ entrants are already successful. The indifference condition (4) simply averages that gain over all possible numbers of successful rivals and multiplies by the success probability q, then compares it to the agent's own cost. Specifically, among the $(n-1)$ rivals that the indifferent agent competes against, $(m-1)$ successful agents can be drawn in $\binom{n-1}{m-1}$ ways with the probability of each draw being $(q\theta_I/\theta_E)^{m-1}(1-q\theta_I/\theta_E)^{n-m}$ and the indifferent agent becomes the mth successful agent with probability q after incurring the investment cost θ_I.

Using Lemma 1, note that only the case $m = 1$ generates a positive contest-stage gain; when two or more agents are successful, all expected contest payoffs are zero, so those terms drop out. Hence we can write

$$\triangle(n,m) = \begin{cases} (\alpha-1)v & \text{if } m = 1 \\ 0 & \text{if } m \geq 2, \end{cases}$$

which reduces (4) to

$$(\alpha-1)vq\left(1-\frac{q\theta_I}{\theta_E}\right)^{n-1} - \theta_I = 0. \tag{5}$$

The first term of (5) is the probability of a single agent being successful in a pool of n entrants multiplied by the increment to the agent's payoff in this case; it is easily verified to be strictly decreasing and strictly convex in θ_I. The second term is the threshold investment cost. There will be two possibilities. The left-hand-side of (5) is positive for all $\theta \leq \theta_E$; in this case, every agent with $\theta \leq \theta_E$ invests, and so we can set the investment threshold $\theta_I = \theta_E$. Otherwise, the indifference condition (5) will have a unique solution $\theta_I \leq \theta_E$, which determines the investment threshold. Thus, the following possibilities can arise in equilibrium.

- **Full investment**: For given θ_E and n, all entrants invest: $\theta_I = \theta_E$.
- **Limited investment**: For given θ_E and n, a subset of entrants invest: $\theta_I < \theta_E$.

The condition for full investment will be determined by the marginal entrant's expected return to investment. Denote the gross expected return to investment (without subtracting the investment cost) of an agent under full investment in an $n-$player contest by $\xi(n)$ where

$$\xi(n) := (\alpha-1)vq(1-q)^{n-1}. \tag{6}$$

With probability q player i succeeds, and with probability $(1-q)^{n-1}$ no one else does, in which case i wins the extra prize $(\alpha-1)v$. It is straightforward to verify that $\xi(n)$ is strictly decreasing in n because more entrants make it less likely that player i is the only successful investor. Proposition 1 documents the equilibrium investment strategy for given θ_E and n.

Proposition 1. *Fix θ_E, n, and suppose that all agents with type $\theta \leq \theta_E \leq 1$ enter. There exists $0 < \theta_I \leq \theta_E$ such that all agents with type $\theta \leq \theta_I$ invest in equilibrium.*

1. If $\theta_E \leq \xi(n)$, then $\theta_I = \theta_E$ so that there is full investment.

2. If $\theta_E > \xi(n)$, then there is limited investment and the investment threshold θ_I uniquely solves (5). The investment threshold θ_I weakly increases in θ_E, v and α. Further, (θ_I/θ_E) strictly decreases in θ_E.

Whether all entrants invest or not depends critically on the value of $\xi(n)$. For $n \geq 2$ entrants, it follows from (6) that $\xi(n)$ is the gross expected return to investment under full investment. The regime of full investment is most likely when $\xi(n)$ is high, i.e. when α (the return to investment) or v (the contest prize) are high, and the number of entrants (n) is low. The effect of the probability of successful investment (q) is ambiguous, since $\xi(n)$ is concave in this parameter, increasing for $q \in (0, 1/n)$, and decreasing thereafter. Relatively low or high values of the success probability depends also on the number of entrants since $\xi(n)$ is maximized at $q = 1/n$. When q is relatively low, it is unlikely

that an investing agent will succeed, but at the same time if it does succeed, it is likely to be alone; in this case this agent expects $\pi_s(n,1) = (\alpha-1)\,v$ at the contest stage by Lemma 1. A relatively high value of q makes it more likely than an investing agent will succeed, but decreases the chances of being the sole successful agent at the contest stage. This decreases the gross expected return of the investment. This nicely demonstrates the interplay between the entry decision and the success probability in determining whether all entrants invest or not.

Limited investment is most likely to occur if the expected return to investment at the contest stage are low, i.e. low α and/or v, a high number of entrants and a probability of investment success that deviates greatly from $q = 1/n$. The gross expected return to investment under limited investment is θ_I, and this is weakly increasing in the prize parameters associated with the contest stage (α, v).

Recall that c is the cost of entry. For given θ_E and n, let $\pi(\theta, n)$ denote the expected payoff of an agent of type θ at the investment stage, given by:

(a) In case of limited investment, i.e., when $\theta_E > \xi(n)$:

$$\pi(\theta,n) = \begin{cases} \theta_I - \theta - c & \text{if } \theta \le \theta_I < \theta_E \\ -c & \text{if } \theta_I < \theta \le \theta_E \\ 0 & \text{if } \theta > \theta_E. \end{cases} \tag{7}$$

(b) In case of full investment, i.e., when $\theta_E \le \xi(n)$:

$$\pi(\theta,n) = \begin{cases} \xi(n) - \theta - c & \text{if } \theta \le \theta_I = \theta_E \\ 0 & \text{if } \theta > \theta_E. \end{cases} \tag{8}$$

For $n = 1$, the sole entrant invests if $\theta \le qv(\alpha-1) = \xi(1)$, and its expected payoff is $v + \xi(1) - \theta - c$ if it invests, and is $v - c$ if it does not invest. Therefore,

$$\pi(\theta,1) = \begin{cases} v + \xi(1) - \theta - c & \text{if } \theta \le \min\{\theta_E, \xi(1)\} \\ v - c & \text{if } \min\{\theta_E, \xi(1)\} < \theta \le \theta_E \\ 0 & \text{if } \theta > \theta_E. \end{cases} \tag{9}$$

For $n = 0$, $\pi(\theta, 0)$ is set to zero.

5. Entry

Consider stage 1. An agent's entry strategy is contingent on its type, which is private information. From (7), (8), and (9), it follows that when there is at least one entrant ($n \ge 1$), the expected payoff of the marginal entrant of type θ_E is

$$\pi(\theta_E, n) = \begin{cases} \begin{cases} \xi(n) - \theta_E - c & \text{if } \theta_E \le \xi(n) \\ -c & \text{if } \theta_E > \xi(n) \end{cases} & \text{if } n \ge 2 \\ \begin{cases} v + \xi(1) - \theta_E - c & \text{if } \theta_E \le \xi(1) \\ v - c & \text{if } \theta_E > \xi(1) \end{cases} & \text{if } n = 1. \end{cases} \tag{10}$$

Consider what the marginal type θ_E can earn by paying c and entering. With probability $\binom{N-1}{n-1}\theta_E^{n-1}(1-\theta_E)^{N-n}$ she faces $n-1$ other entrants, and conditional on that, her continuation payoff is $\pi(\theta_E, n)$ from (7)–(10). Averaging these scenarios is denoted $\mathbb{E}_{n-1}[\pi(\theta_E, n)]$. The cutoff θ_E solves this expected continuation payoff equal to zero:

$$\mathbb{E}_{n-1}\left[\pi(\theta_E, n)\right] = \sum_{n-1=0}^{N-1} \binom{N-1}{n-1} (\theta_E)^{n-1} (1-\theta_E)^{N-n} \pi(\theta_E, n) = 0, \tag{11}$$

where $n - 1 \backsim \text{Binomial}(N-1, \theta_E)$.

Proposition 2 shows that every agent will adopt a threshold strategy and we formally prove that $\mathbb{E}_{n-1}\left[\pi(\theta_E, n)\right]$ is decreasing in θ_E. This observation implies that two possibilities may arise. First, $\mathbb{E}_{n-1}\left[\pi(\theta_E, n)\right]$ is always positive for all $\theta \le 1$; in this case, all types enter, and we can set the entry threshold $\theta_E = 1$. Second, the entry-indifference condition (11) has a unique solution at $\theta_E < 1$, which determines the entry threshold. Thus, we observe the following two possible regimes in equilibrium:

- **Full entry**: All types of agents enter: $\theta_E = 1$.
- **Limited entry**: A subset of agents enters: $\theta_E < 1$.

The expected payoff of the agent of type $\theta = 1$ determines the condition for full entry. Observe that

$$\mathbb{E}_{n-1}[\pi(1,n)] = \pi(1,N) = \begin{cases} \xi(N) - 1 - c & \text{if } 1 \le \xi(N) \\ -c & \text{if } 1 > \xi(N) \end{cases}$$
$$= \underline{c}(N) - c, \tag{12}$$

where $\underline{c}(N) := \max\{\xi(N) - 1, 0\}$ is the expected gross return of the agent with type $\theta = 1$ under full investment and full entry. The following proposition formally characterizes the equilibrium entry strategy.

Proposition 2. *Fix N. There exists $0 < \theta_E \le 1$ such that all agents with type $\theta \le \theta_E$ enter.*

1. If $c \le \underline{c}(N)$, then $\theta_E = 1$ so that there is full entry.

2. If $c > \underline{c}(N)$, then there is limited entry and the entry threshold θ_E uniquely solves (11). Further, θ_E increases in v and α.

From the entry and investment thresholds, we can fully characterize the distribution of the number of entrants and the number of agents who succeed with an investment. Specifically, n follows $Binomial\left(N, \theta_E\right)$ and m follows $Binomial\left(n, q\theta_I/\theta_E\right)$.

Full entry is most likely to occur when $\xi(N)$ is large and/or the entry cost (c) is small; the former occurs when the rewards from the contest stage (α, v) are large and the probability of investment success is at an intermediate value (i.e. close to $q = 1/N$). A straightforward implication of Proposition 2 is that for a contest with an entry fee, full investment occurs whenever there is full entry. This follows from the following observations. First, if $0 < c \le \underline{c}(N)$, then $\underline{c}(N) > 0$ and hence $n = N$. Further, $\underline{c}(N) > 0$ implies that $\theta_E \le 1 < \xi(N)$ and by Proposition 1, we get $\theta_I = \theta_E = 1$, summed up in Corollary 2:

Corollary 2. *If $0 < c \le \underline{c}(N)$, then $\theta_I = \theta_E = 1$.*

The intuition is as follows. If the full entry condition is satisfied, the marginal entrant faces no uncertainty about the number of entrants. She will always compete against the remaining $N-1$ players in an all-pay auction. Since a player that does not realize a successful investment receives zero payoff in the all-pay auction, the marginal entrant can never recover its entry cost by not investing. Therefore, in the full-entry regime, the marginal entrant, and consequently every entrant, is committed to invest.

Limited entry implies $n \le N$, occurring when $c > \underline{c}(N)$, and the investment threshold depends on the realized value of n. The marginal entrant faces uncertainty regarding the number of entrants at the entry stage. If it faces no competition upon entry, which happens when it is the sole entrant, it expects to receive a positive payoff even if it has not realized a successful investment. After entering, the entry cost becomes a sunk cost, and the agent will only invest if the potential gain from investing is sufficient to recover the investment cost. Limited investment can be observed alongside limited entry if $\xi(n)$ falls below the entry threshold θ_E for any given n. Because $\xi(n)$ is decreasing in n, an agent's incentive to invest is reduced with the number of entrants.

Having presented our model and its solution, it is worth discussing some of our modeling assumptions. First, we note that our normalization $\theta \sim U[0,1]$ is without loss of generality. For any atomless type distribution F on $[\underline{\theta}, \bar{\theta}]$, define $u := F(\theta) \in [0,1]$ and write $\tilde{\theta}(u) := F^{-1}(u)$. Then

$$\Pr[\theta \le \theta_I \mid \theta \le \theta_E] = \frac{F(\theta_I)}{F(\theta_E)} = \frac{u_I}{u_E},$$

so the investment condition (5) becomes

$$(\alpha-1)vq\left(1 - q\frac{u_I}{u_E}\right)^{n-1} - \tilde{\theta}(u_I) = 0,$$

and the entry condition (11) averages $\pi(\cdot, n)$ with respect to Binomial $(N-1, u_E)$ instead of Binomial$(N-1, \theta_E)$. All comparative statics in (v, α, q, N, c) are unaffected; results where "1" appears (e.g. Eq. (12)) refer to the top quantile $u = 1$.

Second, our contest analysis in Section 3 assumes that (n, m) are public information at Stage 3. If instead contestants observe only their own success and hold beliefs about the success of rivals, the contest becomes an all-pay auction with binary private values $\{v, \alpha v\}$. In that environment, the zero-payoff results used in Lemma 1((i)–(ii)) no longer apply: a successful type earns positive information rents in expectation even when others may also be successful. Consequently, the investment gain $\Delta(n, m)$ in (4) is replaced by an ex-ante term $\widetilde{\Delta}_n > 0$ that averages over beliefs, so (5) and (6) cease to hold in closed form. The equilibrium entry and investment strategies remain threshold-based, and the qualitative forces we emphasize (investment is driven by the prospect of an advantage at the contest stage; crowding reduces this prospect) continue to operate. However, the expressions become less tractable and comparative statics require a separate analysis. We keep (n, m) observable to preserve transparency and the closed-form characterization.

The interplay between the uncertainty that arises from entry and that from investment is complex. For a contest with an entry fee, full entry implies full investment, and that is a clean result. Limited entry can give rise to both full investment and limited investment in equilibrium. We explore this further in the next section.

6. Comparative statics

In this section, we discuss the comparative statics effects of some key parameters of our model on entry and investment incentives.

6.1. Likelihood of successful investment

The likelihood of success q has contrasting effects on incentives for investment and entry. When q is high, an agent is more likely to succeed in investment, which favorably affects the payoff from entry. However, this agent also anticipates competing against more agents that have made a successful investment, which has a dampening effect. Typically, both entry and investment incentives are high at an intermediate range of q, and this range is concentrated around $q = 1/N$.

To see why, let us examine the full-entry condition: $c \leq \underline{c}(N)$. Observe that $\xi(N)$ is concave for $q \in [0, 1]$, is equal to zero at $q = \{0, 1\}$, and it increases with q for $q < 1/N$, but decreases thereafter. Therefore, the full-entry condition can only be satisfied at an intermediate level of q. We can hence find an interval $\left[\underline{q}, \overline{q}\right], 0 < \underline{q} \leq \overline{q} < 1$ such that $\left[\underline{q}, \overline{q}\right] = \left\{q \in [0, 1] : c \leq \underline{c}(N)\right\}$. Further, this interval can be vacuous if $\max_{q \in [0,1]} \xi(N) < 1 + c$, which holds if $(N-1)^{N-1}/N^N < (1 + c)/v(\alpha - 1)$. We state this formally:

Proposition 3. *Fix $N \geq 2$.*

1. If $(N-1)^{N-1}/N^N < (1 + c)/v(\alpha - 1)$, there is limited entry in equilibrium for every $q \in [0, 1]$.

2. If $(N-1)^{N-1}/N^N \geq (1 + c)/v(\alpha - 1)$, there exist $0 < \underline{q} \leq 1/N \leq \overline{q} < 1$ such that for $q \in \left[\underline{q}, \overline{q}\right]$, there is full entry in equilibrium.

Part 2 indicates that when the contest is sufficiently favorable (high potential prize value, a low entry cost and few potential competitors), all agents will enter if the success probability balances the positive effect of achieving the advantage with the negative one of meeting potentially strong rivals. When there is full entry, there is also full investment. When the full-entry condition does not hold, the full-investment condition is given by $\theta_E \leq \xi(n)$, where n is the realized number of entrants. For given θ_E and n, it easily follows from the shape of $\xi(n)$ that the full-investment condition, if satisfied, only occurs at an intermediate level of q.

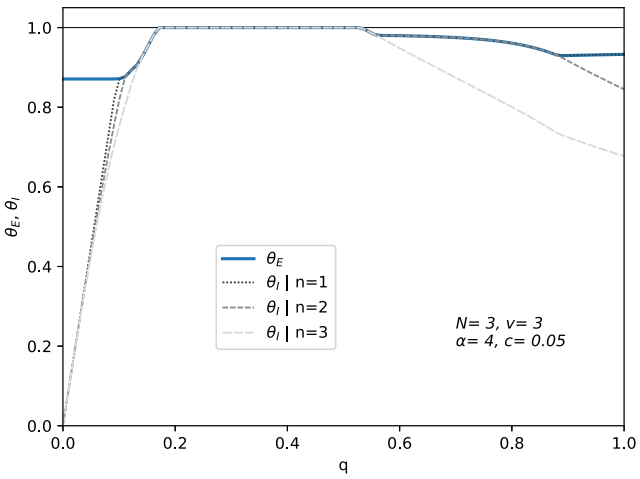

Fig. 1. θ_E and θ_I against q for $N = 3$.

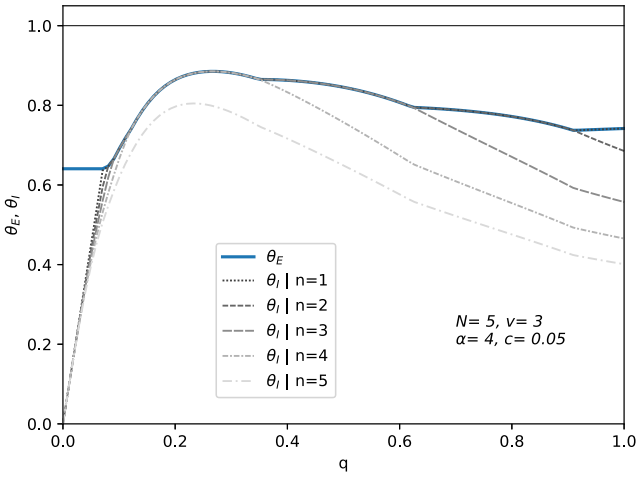

Fig. 2. θ_E and θ_I against q for $N = 5$.

Part 1 shows that when the contest is expected to be less favorable, then no value of the investment success parameter will entice all agents to enter. This is intuitively straightforward. However, it is less obvious how the thresholds θ_E and θ_I change in relation to q in regimes with limited entry and limited investment.

Figs. 1 and 2 illustrate how the two thresholds move against q. In Fig. 1, which plots θ_E and θ_I against q for $N = 3$, there is full entry in equilibrium when $q \in [0.169, 0.531]$. Fig. 2 plots the threshold for $N = 5$ and there is limited entry for every q. These figures also plot θ_I, contingent on $n \in \{1, 2, \ldots, N\}$. The investment threshold θ_I depends on the realized number of entrants. For $q \in [0.169, 0.531]$, $\xi(3) > 1 + c$, and we have full entry in equilibrium. However, since $\xi(5) < 1 + c$, for $N = 5$, there is limited entry in equilibrium for every q.

6.2. Number of agents

The entry incentive weakly diminishes as the total number of agents N increases; this follows from two observations. Firstly, the full-entry condition is satisfied for sufficiently small values of N. Additionally, in cases of limited entry, the entry threshold decreases as N increases. The following proposition documents formally how the entry threshold changes with respect to N.

Proposition 4. *Define* $\bar{N} := \max\{0, 1 + \lfloor(\ln(1 + c) - \ln((\alpha - 1)vq))/\ln(1 - q)\rfloor\}$,

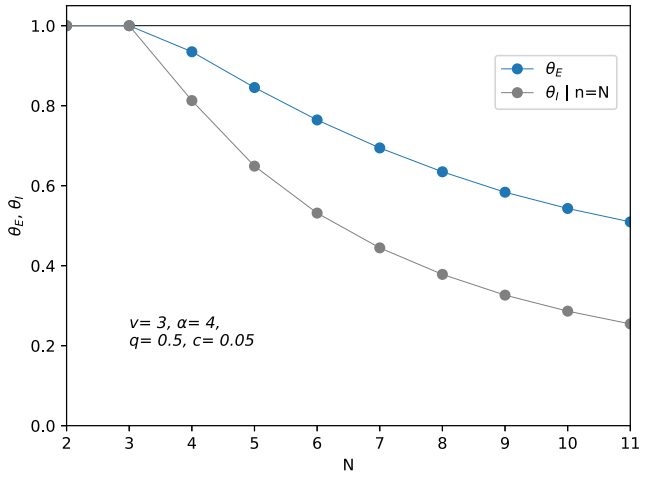

Fig. 3. θ_E and $\theta_I\,|_{n=N}$ against N.

where $\lfloor x \rfloor$ *is the largest integer less than or equal to* x.

1. *For* $N \leq \bar{N}$, $\theta_E(N) = 1$.
2. *For* $N > \bar{N}$, $\theta_E(N)$ *weakly decreases in* N.

Part 1 is the familiar full-entry condition. Since $N \geq 2$, full entry cannot occur if $\bar{N} < 2$, which happens if then entry cost is very large: $1 + c > \xi(2)$; in this case, two agents will not both find it profitable to enter, and full entry will certainly not occur for additional agents. Entry is less attractive the more agents there are. To get the intuition behind the result, consider from the perspective of the marginal entrant when there are N players. The marginal entrant expects a positive payoff in two scenarios. First, it might be the only entrant, and the expected post-entry payoff in this event is $v + \max\{0, \xi(1) - \theta_E\}$. Second, there could be $(n-1)$ other entrants for various values of n, and the marginal entrant's expected post-entry payoffs in these events are $\max\{0, \xi(n) - \theta_E\}$. When the number of players increases by 1, the likelihood of the first scenario decreases, and the payoffs associated with the second scenario decrease for every n. Consequently, the marginal entrant's expected post-entry payoff declines as N increases.

Describing the impact of N on the investment threshold is more complex because the threshold depends on the realized number of entrants, and the distribution of the number of entrants changes as N moves. If we fix the number of entrants at a given n and examine how changing N affects the investment threshold, we can infer from Proposition 1 that θ_I will also decrease. This is because the two thresholds are positively related.

Fig. 3 numerically illustrates the relationship between the entry threshold (shown as the blue-colored curve) and N. We also plot the investment threshold (shown as the gray-colored curve) when all agents have entered ($n = N$); in this case the investment threshold solves:

$$(\alpha - 1)\, vq \left(1 - \frac{q\theta_I}{\theta_E(N)}\right)^{N-1} - \theta_I = 0. \tag{13}$$

As N increases, it affects θ_I in two ways: first, by directly influencing the threshold that solves (13), and second, by decreasing the entry threshold. Proposition 1 implies that the second effect leads to a decline of the investment threshold. Furthermore, it can be shown that the direct effect of N decreases θ_I.[20]

[20] The proof of this follows the technique used to show part 2 of Proposition 4, and is omitted here.

6.3. Entry fee

An entry fee adversely affects the entry incentive. As with our analysis of the effects of N, we can illustrate the dampening effect of c with two observations.

Firstly, the full-entry condition is satisfied only for sufficiently small values of c, specifically, for $c \leq \underline{c}(N)$. In addition, when there is limited entry, we can examine how θ_E moves with respect to c by analyzing (11). Proposition 5 documents the effect of c on θ_E.

Proposition 5. *For* $c \leq \underline{c}(N)$, $\theta_E = 1$. *For* $c > \underline{c}(N)$, θ_E *is strictly decreasing in* c.

The mechanism behind this result is straightforward: entry fees directly reduce the marginal entrant's payoff in all possible scenarios, thereby dampening the incentive to enter.

The monotone relationship between the entry fee and the entry threshold has important implications for design problems. A contest designer can achieve her desired entry threshold by adjusting the entry fee. For a given θ_E, let $\hat{c}(\theta_E)$ be the maximum level of entry cost that results in an entry threshold equal to θ_E. Then we can prove the following proposition.

Proposition 6. *Any entry threshold* $\theta_E \in [0, 1]$ *can be implemented by choosing an entry fee* $c = \hat{c}(\theta_E)$. *Furthermore,* $\hat{c}(\theta_E)$ *is continuous and differentiable.*

7. Contest design

Contest models often purport the existence of a designer that sets various instruments in the competition in order to achieve some objective. In this section, we consider three different objectives, exploring the interplay between entry, investment and contest effort.

7.1. Expected effort

The most common objective in the contest literature is to maximize expected effort. In our model, this is a complex calculation since each rival takes up to three decisions that affect final effort (entry, investment and effort). The resolution of the entry and investment decisions affects the number and composition of the rivals that meet in the ultimate contest. Under the symmetric cutoff equilibrium, the number of entrants is $n \sim \text{Binomial}(N, \theta_E)$, and conditional on n, the number of successful investors is $m \sim \text{Binomial}(n, \kappa(n))$, where

$$\kappa(n) = \begin{cases} q, & \text{if } \theta_E \leq \xi(n) \quad \text{(full investment)} \\ q\,\theta_I(n)/\theta_E, & \text{if } \theta_E > \xi(n) \quad \text{(limited investment)} \end{cases} \tag{14}$$

and $\theta_I(n)$ is the unique solution to the investment indifference condition in (5).

Recall that $T(n, m)$ is the total expected contest effort from Lemma 1:

$$T(n, 0) = v, \qquad T(n, m) = \alpha v \text{ for } m \geq 2, \qquad T(n, 1) \in \left[T^{\min}(n, 1),\, T^{\max}(n, 1)\right],$$

and $T(1, m) = 0$. If there is a single successful player, then there are a continuum of equilibria at the contest stage, and we require an equilibrium selection rule to tie down the effort. For any selection rule $T^{\star}(n, 1) \in [T^{\min}(n, 1),\, T^{\max}(n, 1)]$, we can write $\tau(n)$ as expected contest efforts when there are n entrants as $\tau(0) = \tau(1) = 0$ and, for $n \geq 2$,

$$\begin{aligned} \tau(n) = v(1 - \kappa(n))^n + \alpha v\Big(1 - (1 - \kappa(n))^n - n\kappa(n)(1 - \kappa(n))^{n-1}\Big) \\ + T^{\star}(n, 1)\, n\kappa(n)(1 - \kappa(n))^{n-1} \end{aligned} \tag{15}$$

With probability $(1 - \kappa(n))^n$, none of the entrants are successful with their investment, leading to effort level v in the contest. The probability of a single successful investor is $n\kappa(n)(1 - \kappa(n))^{n-1}$ in which the selection rule gives expected effort as $T^{\star}(n, 1)$ at the contest stage. The rest of the probability mass falls on cases in which as least two players

have successful investments, given contest effort αv. From Corollary 1, maximizing expected effort for any number of players involves moving probability mass to cases where $m \geq 2$, and reducing the likelihood that a single investor succeeds.

Equivalently, for $n \geq 2$ we can write (15) as:

$$\tau(n) = \alpha v - (\alpha v - v)(1-\kappa(n))^n - (\alpha v - T^\star(n,1))\, n\kappa(n)(1-\kappa(n))^{n-1}. \quad (16)$$

This shows that expected contest effort is equal to its highest level (with at least two successful investors), modified by the amount of effort lost in case no one succeeds, and the amount lost if a single player has a successful investment. It is straightforward to show that the expected effort conditional on n entrants is increasing in the per-entrant success rate: $\partial\tau(n)/\partial\kappa(n) > 0$.

Then, by averaging over the expected number of entrants, total expected effort is:

$$\mathbb{E}[T] = \sum_{n=0}^{N} \binom{N}{n} \theta_E^n (1-\theta_E)^{N-n}\, \tau(n). \quad (17)$$

While the exact expected effort will depend on the equilibrium selection rule used, we can provide tight bounds on this by choosing $T^{\min}(n,1)$ and $T^{\max}(n,1)$.

A designer can increase $\mathbb{E}[T]$ by raising the prize spread α and/or the basic prize v directly. Increasing entry can have countervailing effects on the total amount of expected contest effort. A higher θ_E means more entrants on average, but the total effect depends crucially on the effect that this has on the total probability of investment success in (14). Under full investment, there is a fixed per-entrant success rate, so that more entry does not affect this. Increasing entrants raises the probability of at least two successes, achieving the highest level of effort; however, if q is small, the probability of a single success also increases with more entry, achieving the lowest contest effort. Even in the case of full investment, maximizing expected contest effort may imply limiting entry.

With limited investment there is an additional effect, since the probability of success is now $\kappa(n) = q\theta_I(n)/\theta_E$, and Proposition 1 shows that the ratio θ_I/θ_E strictly decreases in θ_E. Hence, increasing θ_E decreases $\kappa(n)$, which shifts probability away from $m \geq 2$, lowering expected effort conditional on n.

Proposition 7 looks explicitly at the case in which there would be limited investment with both N and $N-1$ entrants, i.e. $1 > \xi(N-1) (> \xi(N))$. Two following corollaries deal with other investment regimes.[21]

Proposition 7. *Suppose a contest designer maximizes total expected contest effort, and that $1 > \xi(N-1)$. Then the left derivative of $\mathbb{E}[T]$ at full entry satisfies*

$$\frac{d}{d\theta_E}\mathbb{E}[T]\Big|_{\theta_E=1} = N\left[\tau(N, q\theta_I(N)) - \tau(N-1, q\theta_I(N-1))\right] + \frac{\partial\tau(N)}{\partial\theta_E}\Big|_{\theta_E=1}. \quad (18)$$

Consequently, there exists $\varepsilon > 0$ such that reducing entry (choosing $\theta_E \in (1-\varepsilon, 1)$) raises expected total effort iff this derivative is negative.

This first term in (18) is a composition effect that works by shifting the Binomial mass over n; the second term reflects the dilution of investment that occurs in limited investment since $\kappa(n)$ falls in θ_E. When $\xi(N) \leq 1$, implying limited investment at full entry, the sign of $\partial\tau(N)/\partial\theta_E|_{\theta_E=1}$ depends on the response of $\kappa(N) = q\,\theta_I/\theta_E$ to changes in θ_E.

A second possibility is that there is full investment with $N-1$ entrants, but limited investment when all N players enter, i.e. $\xi(N) < 1 < \xi(N-1)$. The fact that the investment incentive falls around $\theta_E = 1$ give an extra incentive to limit entry as outlined in the following corollary.

Corollary 3. *If $\xi(N) < 1 \leq \xi(N-1)$, then $\kappa(N) = q\,\theta_I/\theta_E < q = \kappa(N-1)$. Then*

$$\frac{d}{d\theta_E}\mathbb{E}[T]\Big|_{\theta_E=1} = N\left[\tau(N;\kappa(N)) - \tau(N-1;q)\right] + \left(\frac{\partial\tau}{\partial\kappa}\Big|_{n=N,\kappa=\kappa(N)}\right)\cdot\left(\frac{\partial\kappa(N)}{\partial\theta_E}\Big|_{\theta_E=1}\right),$$

where $\frac{\partial\tau}{\partial\kappa} > 0$ for $n \geq 2$ and $\frac{\partial\kappa(N)}{\partial\theta_E}\Big|_{\theta_E=1} < 0$ under limited investment. Hence the second term is strictly negative, reinforcing any incentive to limit entry coming from the first term.

The final possibility occurs when $\xi(N) > 1$ so that there is full investment with $N-1$ and N entrants. Corollary 4 gives a simple condition which implies that limiting entry will increase expected contest effort.

Corollary 4. *If $\xi(N) > 1$, then $\theta_I = \theta_E$ for all $\theta_E \in [0,1]$ and hence $\kappa(N) \equiv q$ around $\theta_E = 1$. Therefore, $\partial\tau(N)/\partial\theta_E|_{\theta_E=1} = 0$ and*

$$\frac{d}{d\theta_E}\mathbb{E}[T]\Big|_{\theta_E=1} = N\left[\tau(N,q) - \tau(N-1,q)\right].$$

In this case, limiting entry raises expected effort iff $\tau(N,\theta_E=1) < \tau(N-1,\theta_E=1)$, evaluated at $\theta_E = 1$.

As noted above, increased entry does not affect the per-entrant success probability in this case. However, especially for low q, the composition of rivals can increase the chance of the least effort outcome at the contest stage, so that a designer would prefer to limit the number of competitors.

7.2. Total fees collected

In many applications, a designer cannot directly monetize the efforts provided at the contest stage. Public good or open intellectual property competitions may involve a commitment by the sponsor to allow any breakthrough to be open or belong to the solvers. In competitive coding environments, for example, the value of effort may accrue to participants as skill-building or signaling. Since an organizer cannot appropriate the human-capital return, the principal can monetize through an entry fee. In other cases, fees may be charged in order to recoup the expenses from running the contest, or to limit participation, especially by low-quality agents. Taylor (1995) notes that the US Federal Communications Commission opened a contest to design the technology standard for HD-TV, charging an entry fee of 200,000 USD. Competitions in music, writing, sports and architecture often charge an entry fee. In 2023 the participation fees for the Eurovision song contest totaled 6.2 million Euros.[22]

The contest designer can extract surplus from the participants by charging an entry fee, c, to maximize the expected value of the total fees received. As the number of entrants n follows *Binomial* (N,θ_E), the expected value of fees received is $c\mathbb{E}(n) = cN\theta_E(c)$ for a given c. The optimal choice of c, therefore, maximizes $Nc\theta_E(c)$.

By replacing c by $\hat{c}(\theta_E)$, we can rewrite the optimization problem as a choice problem over the possible entry threshold values. The designer's preferred choice of θ_E maximizes the expected value of fees collected, denoted by V_f:

$$V_f(\theta_E) := N\hat{c}(\theta_E)\,\theta_E. \quad (19)$$

Because of continuity and differentiability of $\hat{c}(\theta_E)$, V_f is continuous and differentiable in θ_E. Therefore, if the optimization problem has an interior solution, this must satisfy the first-order necessary condition:

$$\hat{c}(\theta_E) + \theta_E \frac{d\hat{c}(\theta_E)}{d\theta_E} = 0.$$

[21] Here we use a second argument in $\tau(n,\cdot)$ to indicate the per-entrant success of investment.

[22] See Eurovision (2024).

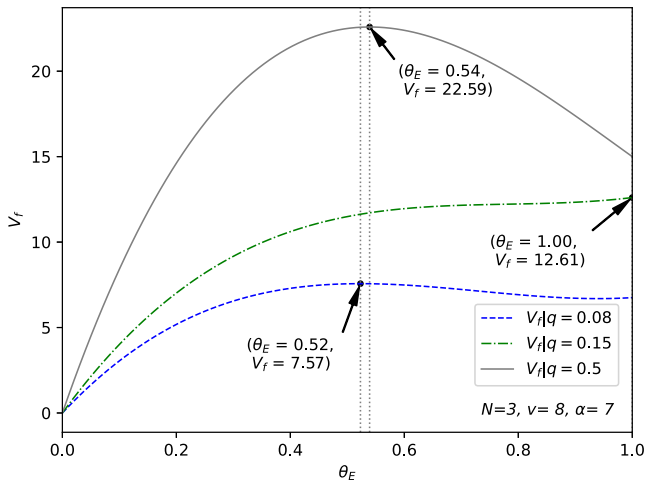

Fig. 4. V_f against θ_E for different q values.

In general, the objective function (19) can exhibit both concave and convex properties. Proposition 8 outlines the sufficient condition under which limited entry is preferred.

Proposition 8. *Consider a contest designer who maximizes the total fees received. If* $\underline{c}(N) = 0$*, then the designer prefers limited entry. If* $\underline{c}(N) > 0$*, then a sufficient condition for the designer to prefer limited entry is given by*

$$\xi(N-1)(1-Nq) - 2 - \upsilon \cdot \mathbf{1}_{\{N=2\}} < 0, \tag{20}$$

where $\mathbf{1}_{\{N=2\}}$ *is an indicator function that takes the value 1 if* $N = 2$*, and 0 otherwise.*

Limited entry is preferred for large N values. This is because $\underline{c}(N) = 0$ for sufficiently high N values. Further, when $\underline{c}(N) > 0$, the sufficient condition (20) is more likely to hold for high values of N: $\xi(N-1)(1-Nq) \leq 0$ for $N \geq 1/q$ and is positive but decreasing in N for $N < 1/q$. Similarly, for sufficiently large q values, limited entry is preferred. Although $\underline{c}(N)$ moves non-monotonically with respect to q, it is decreasing in q for $q \geq 1/N$, and the sufficient condition is always negative for $q \geq 1/N$. It is important to note that if $dV_f/d\theta_E > 0$ as θ_E approaches 1, we cannot definitely conclude that full entry is preferred, as there could be a local interior maximum even if V_f is increasing at $\theta_E = 1$. This is illustrated in Fig. 4.

Fig. 4 depicts how V_f changes with respect to θ_E under various scenarios. We set $N = 3$, $\upsilon = 8$, $\alpha = 7$, and vary q within the set $\{0.08, 0.15, 0.5\}$. In all these scenarios, $\underline{c}(N) > 0$. The sufficiency condition in (20) is met when $q = 0.5$, but it is not satisfied for $q = 0.08$ and $q = 0.15$. For $q = 0.5$, V_f (represented by the continuous curve) reaches its maximum value of 22.59 at $\theta_E = 0.54$. For $q = 0.15$, V_f (represented by the green dot-dashed curve) achieves its maximum at the boundary $\theta_E = 1$. For $q = 0.08$, V_f (shown as the blue dashed curve) attains its interior maximum at $\theta_E = 0.52$.

7.3. The expected number of investors

Innovation contests often involve a trade-off between admitting many participants who consequently supply low effort, but who may increase the chance that a valuable innovation is made (Terwiesch and Xu, 2008; Boudreau et al., 2011). Pan et al. (2017) analyze 29 health-focused innovation design contests, finding that two types of contest emerged. One focused on the process of mass community engagement, and the other on high-quality, innovative outcomes. In the latter, it might be a sensible objective in our model to maximize the expected number of investors.

The likelihood of investment by a player, conditional upon entry, is θ_I/θ_E. As the number of entrants follows Binomial (N, θ_E), the expected number of investors, denoted by V_{inv}, is given by

$$V_{inv}(\theta_E) = \sum_{n=0}^{N} \binom{N}{n} (\theta_E)^n (1-\theta_E)^{N-n} \frac{n\theta_I(n)}{\theta_E}, \tag{21}$$

where $\theta_I(n)$ and θ_E satisfy the conditions described in Propositions 1 and 2. After rearranging terms, (21) can be simplified as

$$V_{inv}(\theta_E) = N\mathbb{E}_{n-1}\left[\theta_I(n)\right], \tag{22}$$

where $(n-1)$ follows Binomial $(N-1, \theta_E)$.

Consider first the case $\underline{c}(N) > 0$, which occurs when $\xi(N) > 1$. Note that if $\theta_E \leq \xi(N)$, then $\theta_E \leq \xi(n)$ for every $n \leq N$, and consequently, $\theta_I(n) = \theta_E$. From (22), $V_{inv}(\theta_E) = N\theta_E$, which is maximized at $\theta_E = 1$. Therefore, the designer prefers full entry.

Next, consider the case when $\underline{c}(N) = 0$, which occurs when $\xi(N) \leq 1$. As we have argued in the previous case, replacing c by $\hat{c}(\theta_E)$ in (22), we can express the designer's problem as a choice problem over the possible values of θ_E. Thus, we can study the derivatives of V_{inv} with respect to θ_E at the boundary values to derive a sufficient condition for the existence of a preferred entry threshold strictly below 1. Proposition 9 documents the sufficient condition under which limited entry is preferred.

Proposition 9. *Consider a contest designer who maximizes the expected number of investors. If* $\underline{c}(N) > 0$*, then the designer prefers full entry. If* $\underline{c}(N) = 0$*, then a sufficient condition for the designer to prefer limited entry is given by*

$$\frac{(N-1)q\hat{\theta}^2}{(N-2)q\hat{\theta}+1} + (N-1)\left(\hat{\theta} - \hat{\hat{\theta}}\right) < 0, \tag{23}$$

where $\hat{\theta} := \lim_{\theta_E \to 1} \theta_I(N)$ *and* $\hat{\hat{\theta}} := \lim_{\theta_E \to 1} \theta_I(N-1)$.

Why might a designer choose to limit entry? In scenarios where full investment occurs (i.e., when $\theta_I = \theta_E$), the designer generally benefits from raising the entry threshold. However, she might consider limiting entry specifically when the investment threshold is significantly lower than the entry threshold for certain values of n. In such cases, the designer's motivation for increasing θ_E is influenced not just by the investment thresholds across different events with varying n values, but also by the rate at which the probabilities of these events shift. Notably, as θ_E approaches 1, the rates of change in probabilities of all events, except when $n = N$ and $n = N-1$, asymptotically approach zero. In contrast, the probabilities of the events of $n = N-1$ and $n = N$ decrease and increase, respectively, as θ_E nears 1. Additionally, given that $\theta_I(N-1)$ is strictly larger than $\theta_I(N)$ (when both are below θ_E), the reduction in probability of $n = N-1$ can sometimes weaken the designer's motivation to raise the entry threshold. The sufficient condition outlined in (23) precisely characterizes such scenarios.

Fig. 5 illustrates how V_{inv} changes with θ_E. We set $N = 3$, $\upsilon = 3$, $\alpha = 4$, and consider cases where q is 0.08 and 0.5. For $q = 0.5$, $\underline{c}(N) = 0.12$, and V_{inv} (depicted by the green dot-dashed curve) achieves its maximum at the boundary $\theta_E = 1$. For $q = 0.08$, $\underline{c}(N) = 0$ and the sufficiency condition in (23) is satisfied; V_{inv} (represented by the blue dashed curve) reaches its maximum at $\theta_E = 0.72$, with the maximum value being 1.95. Notably, for $q = 0.08$, V_{inv} at $\theta_E = 1$ is marginally lower, measured at 1.94.

8. Conclusion

This paper extends the extant contest literature by considering two decisions – entry and investment – made in pre-contest play. A common assumption is that a fixed set of contestants meet to fight for a winner-take-all prize. We modify this to make the set and composition of rivals the result of conscious choices made by rivals before the

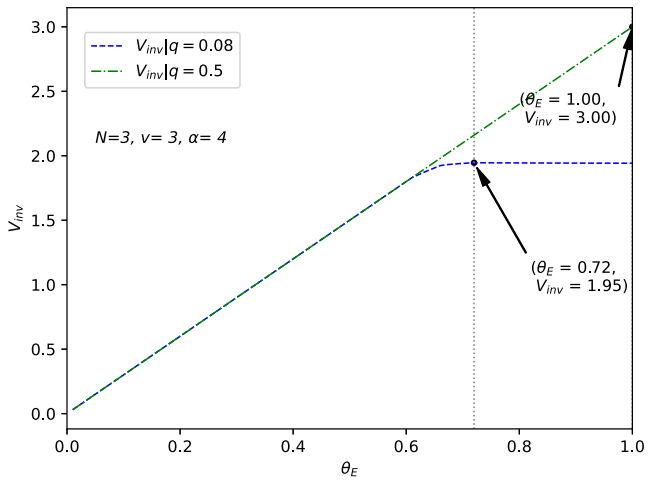

Fig. 5. V_{inv} against θ_E for different q values.

contest. Pre-contest actions are also costly, so that ignoring this may lead to theoretically predicted contest effort in excess of that actually observed.

While others have considered either the entry or the investment decision, our contribution looks at their interaction. A player's payoff at the contest stage, is dependent on the number and quality of rivals. We uncover a complex interplay between entry and investment decisions, captured by some simple parameters in the analysis. While the potential size of the prize and the cost of entering the contest have predictable effects on entry and investment, the probability of a successful investment has a non-monotonic effect. Furthermore, the region of non-monotonicity is inextricably linked to the total number of competitors. The threshold values of investment cost that determine entry and investment in equilibrium resolve the complex decision making process. Agents know that if they meet homogeneous rivals in the final contest (whether they are all successful or unsuccessful in their investment), they will compete away a large proportion of the contested prize, lowering the incentive to enter and make a pre-contest investment. Meeting one or more stronger rivals in the contest also leads to a low expected payoff and weak incentives. The driving force behind entry/investment is the promise of being the lone strong agent in the contest, who is guaranteed a large prize for low effort. If the probability of achieving a successful investment is very low, then an agent is likely to meet equally weak rivals in the upcoming contest. If the success probability is high, an agent will expect to meet several equally strong rivals. An intermediate probability of success balances these two scenarios, making entry and investment more attractive.

We have shown that any entry threshold can be implemented by appropriate setting of the cost of entering the contest. A sufficiently low (but positive) fee can entice full entry into the contest, and this in turns guarantees that all entrants invest. A higher entry cost discourages entry by those who have the highest marginal cost of investment. It may well still be the case that all entrants invest also in this scenario. All other things equal, a large initial number of competitors weakens the incentive for both entry and investment.

To highlight our contribution, it is instructive to consider what would happen by just considering one of the pre-contest stages, rather than the interaction of both. Suppose there were no investment possibility. Then the contest is between a set of homogeneous competitors, and it is well known from Lemma 1 that there will be a continuum of equilibria in which two rivals fight and compete away the whole value of the prize. If entry is costly, no one will enter. Suppose that there is no entry decision (or entry cost), so that all players make an investment choice. Our analysis of the investment stage is still valid given a fixed number of competitors, replacing the entry threshold by 1. Only the uncertainty over the return is now important for the investment decision. Full (partial) investment occurs if the gross expected return, given the fixed number of rivals, is greater (smaller) than 1, the investment cost of the least efficient player. The non-monotonic relationship between the investment threshold and the probability of success will still hold, and this part of the model adequately captures behavior in a market with no active entry decision. In other applications, it is necessary to undertake a costly action in order to be allowed into the contest (such as military procurement mentioned in the introduction). Our full analysis also captures these markets, showing that the entry decision can effectively screen participants, imparting important information about rivals at the investment stage.

CRediT authorship contribution statement

Derek J. Clark: Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Tapas Kundu:** Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Disclosure statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We would like to thank two anonymous referees for helpful comments on a previous draft. We also thank the seminar participants at the 10th Annual Conference on Contests: Theory and Evidence (University of Reading, UK), the 18th Annual Conference on Economic Growth and Development (Indian Statistical Institute, New Delhi, India), and our colleagues at the School of Business and Economics, UiT, Norway, and Oslo Business School, Oslo Metropolitan University, Norway, for their helpful comments. All remaining errors are our own.

Appendix A

Appendix A contains the proofs that are omitted in the main text. We will begin by stating two well-known results that are useful in proving our main findings. These are monotonicity results for binomial averages. Lemma A.1 follows directly from Chapter 4 in Wolfstetter (1999), or by combining Theorem 1.1a in Klenke and Mattner (2010) with results in Section 1.A in Shaked and Shanthikumar (2007). Lemma A.2 combines Theorem 1.1a in Klenke and Mattner (2010) with Theorems 1.A.1 and 1.A.3(a) in Shaked and Shanthikumar (2007).

Lemma A.1. *Consider a function $f(p,X) : [0,1]\times\mathbb{N} \to \mathbb{R}$ that is decreasing in both arguments, p and X. Let $m \geq 2$ be an integer and let X follow Binomial(m,p). Then, $d\mathbb{E}_X[f(p,X)]/dp \leq 0$.*

Furthermore, if f is strictly decreasing in X for some $X \in \{0,1,\ldots,m\}$, or if f is strictly decreasing in p at some $X \in \{0,1,\ldots,m\}$, then $d\mathbb{E}_X[f(p,X)]/dp < 0$.

Lemma A.2. *Consider a function $f(X) : \mathbb{N} \to \mathbb{R}$ that is decreasing in X. Fix $p \in [0,1]$ and define a function $F : \mathbb{N} \to \mathbb{R}$ by $F(m) = \mathbb{E}_X[f(X)]$ where $X \backsim$ Binomial(m,p). Then, $F(m+1) \leq F(m)$. Furthermore, the inequality holds strictly if f is strictly decreasing for some $X \in \{0,1,\ldots,m\}$.*

Proof of Lemma 1. Parts (i) and (ii) follow directly from Baye et al. (1996, Theorem 1). Part (iii) uses their Theorem 2. Denoting the expected effort of the skilled agent by e_s, we can use Baye et al. (1996, Theorem 2C) to write the expected sum of efforts as

$$T(n,1) = \sum_{i=1}^{n} Ex_i = \frac{v}{\alpha} + \left(1 - \frac{1}{\alpha}\right)\mathbb{E}x_s, \tag{A.1}$$

where $\mathbb{E}x_s$ is the expected effort of the single skilled agent, and this varies across the continuum of equilibria. Denoting the mixed strategy of the skilled agent by $G_s(x_s), x_s \in \left[\underline{x_s}, \overline{x_s}\right]$, we have

$$\mathbb{E}x_s = \int_{\underline{x_s}}^{\overline{x_s}} (1 - G_s(x_s)dx_s. \tag{A.2}$$

In the equilibrium leading to the least effort, we use Baye et al. (1996, Eq. (4)) to find the mixed strategy of the skilled agent as

$$G_s(x_s) = \frac{x_s}{v}\left(1 - \frac{1}{\alpha} + \frac{x_s}{\alpha v}\right)^{\frac{2-n}{n-1}}, x_s \in [0, v]. \tag{A.3}$$

Inserting (A.3) into (A.2) and then into (A.1) gives $T^{min}(n,1)$ after some rearrangement.

Further, when only one unskilled agent is active, Baye et al. (1996, Eq. (4)) implies

$$G_s(x_s) = \frac{x_s}{v}, x_s \in [0, v]. \tag{A.4}$$

Inserting (A.4) into (A.2) and into (A.1) gives $\mathrm{T}^{max}(n,1)$. It is straightforward to verify by substitution that $T^{min}(2,1) = T^{max}(2,1)$. □

Proof of Proposition 1. Consider that agents are following a threshold entry strategy: all types less than θ_E enter. An agent's return to investment $q \triangle (n,m) - \theta$ is decreasing in its investment cost θ, implying that its investment strategy follows a cutoff rule as well. Further, because all agents have the same entry cost c, the investment cutoff will also be the same across all agents who enter. Denoting the investment threshold by θ_I, the probability that a randomly picked agent would have a successful investment conditional on entry is $q\Pr\left[\theta \le \theta_I\right] / \Pr\left[\theta \le \theta_E\right] = q\theta_I/\theta_E$. Since agents' success are independent events, the probability that an agent faces exactly $m-1$ successful agents out of $n-1$ entrants is given by $\binom{n-1}{m-1}\left(q\theta_I/\theta_E\right)^{m-1}\left(1 - \left(q\theta_I/\theta_E\right)\right)^{n-m}, m-1 \in \{0, \ldots, n-1\}$. Therefore, the expected return to investment is

$$q\left[\sum_{m-1=0}^{n-1}\binom{n-1}{m-1}\left(\frac{q\theta_I}{\theta_E}\right)^{m-1}\left(1 - \frac{q\theta_I}{\theta_E}\right)^{n-m} \triangle (n,m)\right] - \theta,$$

which further reduces to $(\alpha-1)vq\left(1 - \left(q\theta_I/\theta_E\right)\right)^{n-1} - \theta$ because $\triangle (n,m) = 0$ for all $m \ge 2$. If the expected return for the marginal entrant is positive, which happens if $(\alpha-1)vq(1-q)^{n-1} - \theta_E \ge 0$, or equivalently, $\xi(n) \ge \theta_E$, then all agents who enter must invest. In this case, $\theta_I = \theta_E$. If the expected return is negative for the agent with type θ_E, which happens if $\xi(n) < \theta_E$, then only a subset of agents must invest, and θ_I uniquely satisfies (5). The uniqueness follows from the fact that the marginal investor's expected return is also decreasing in θ_I.

Let $\Omega := q\left(1 - \left(q\theta_I/\theta_E\right)\right)^{n-1} v(\alpha-1)$. From (5) we can find

$$\frac{d\theta_I}{d\theta_E} = \frac{\frac{\partial\Omega}{\partial\theta_E}}{1 - \frac{\partial\Omega}{\partial\theta_I}} = \frac{q\theta_I(n-1)\Omega}{\theta_E\left(\theta_E - q\theta_I + q(n-1)\Omega\right)} > 0.$$

The positive marginal effects of v and α can also be derived similarly. Furthermore,

$$\frac{\partial}{\partial\theta_E}\left(\frac{\theta_I}{\theta_E}\right) = \frac{\theta_E\frac{d\theta_I}{d\theta_E} - \theta_I}{\theta_E^2} = \frac{-\theta_I(\theta_E - q\theta_I)}{\theta_E^2\left(\theta_E - q\theta_I + q(n-1)\Omega\right)} < 0. \quad \square$$

Proof of Proposition 2. Fix the opponents' strategy at cutoff θ_E and define the continuation payoff from entering at type θ as

$$\tilde{\pi}(\theta,n) := \begin{cases} \theta_I - \theta - c, & \text{if } \theta_E > \xi(n) \text{ and } \theta \le \theta_I, \\ \xi(n) - \theta - c, & \text{if } \theta_E \le \xi(n) \text{ and } \theta \le \theta_E, \\ -c, & \text{if } n \ge 2 \text{ and the agent does not invest,} \\ v - c, & \text{if } n = 1 \text{ and the agent does not invest,} \end{cases}$$

where $\theta_I = \theta_E$ under full investment and otherwise solves (5). By (7)–(9), each branch is weakly decreasing in θ (with slope -1 in the investing region, flat otherwise). Let

$$U(\theta) := \sum_{n=1}^{N}\binom{N-1}{n-1}\theta_E^{n-1}(1-\theta_E)^{N-n}\tilde{\pi}(\theta,n)$$

be the expected continuation payoff from entering. Then $U(\theta)$ is weakly decreasing in θ, implying that a cutoff entry strategy is optimal. Since all agents face the same entry cost, the threshold is the same for all of them. We denote this threshold as θ_E which satisfies $U(\theta_E) = 0$, which is (11). For any $\theta_1 > \theta_E$, monotonicity gives $U(\theta_1) \le U(\theta_E) = 0$. Hence a higher type has non-positive expected payoff from entering when others use the cutoff θ_E, so deviating from entry is not profitable.

At the entry stage, the expected payoff of the agent of type θ_E is $\mathbb{E}_{n-1}\left[\pi\left(\theta_E, n\right)\right]$ where the number of other players, $n-1$, is a random variable following a Binomial distribution with parameters N and θ_E. It directly follows from Lemma A.1 that $\mathbb{E}_{n-1}\left[\pi\left(\theta_E, n\right)\right]$ is decreasing in θ_E. The full-entry condition can therefore be derived from the expected payoff of the agent of type $\theta = 1$, which is given by $\mathbb{E}_{n-1}\left[\pi(1,n)\right] = \pi(1,N) = \underline{c}(N) - c$. Therefore, if $c \le \underline{c}(N)$, every agent has an incentive to enter, and $\theta_E = 1$. On the other hand, if $c > \underline{c}(N)$, $\pi(1,N)$ is negative and (11) has a unique solution determining the entry threshold.

Further, considering $\mathbb{E}_{n-1}\left[\pi\left(\theta_E, n\right)\right]$ as a function $G\left(\theta_E, z\right)$ of θ_E and a generic parameter z, we can work with the total derivative of (11) to get

$$\frac{d\theta_E}{dz} = -\frac{\partial G/\partial z}{\partial G/\partial\theta_E}.$$

As $\partial G/\partial\theta_E \le 0$, $d\theta_E/dz$ has the same sign as $\partial G/\partial z$, whenever both terms are well-defined. Applying this observation and the fact that $\pi\left(\theta_E, n\right)$ is increasing in v and α, we conclude that θ_E increases in v and α. □

Proof of Proposition 3. It follows from Proposition 2 that there is limited entry if $c > \underline{c}(N)$, or equivalently, if $\xi(N) < 1 + c$, which holds if

$$f_1(q) := q(1-q)^{(N-1)} < \frac{1+c}{v(\alpha-1)}.$$

Examining the first derivative, we get that f_1 is increasing in $q \le 1/N$, and decreasing thereafter, implying

$$\max_{q\in[0,1]} f_1(q) = \frac{(N-1)^{N-1}}{N^N}.$$

If $\max_{q\in[0,1]} f_1(q) < (1+c)/v(\alpha-1)$, there is limited entry for every $q \in [0,1]$, which proves part (i) of the proposition. Further, if $\max_{q\in[0,1]} f_1(q) \ge (1+c)/v(\alpha-1)$, then there will be full entry for some q. Given that f_1 is increasing up to $1/N$ and decreasing thereafter, $f_1(q)$ must be higher than $(1+c)/v(\alpha-1)$ at an interval $\left[\underline{q}, \overline{q}\right]$, containing $1/N$. □

Proof of Proposition 4. Observe that the full-entry condition $c \le \underline{c}(N)$ can be rewritten as

$$(1+c) \le \xi(N) \Leftrightarrow (1-q)^{N-1} \ge \frac{(1+c)}{vq(\alpha-1)}.$$

By taking the logarithm on both sides and noting that $\ln(1-q)$ is negative, we can express the above inequality as

$$N \le 1 + \frac{\ln\left((1+c)/(vq(\alpha-1))\right)}{\ln(1-q)}.$$

Defining $\bar{N}$ as $\max\{0, 1 + \lfloor(\ln(1+c) - \ln((\alpha-1)vq))/\ln(1-q)\rfloor\}$, where $\lfloor x\rfloor$ is the largest integer less than or equal to x, part (1) of the proposition directly follows.

Next, suppose $N > \bar{N}$, in which case, there is limited entry and θ_E satisfies (11). We express $\mathbb{E}_{n-1}\left[\pi\left(\theta_E, n\right)\right]$, where $(n-1)$ follows the distribution Binomial $\left(N-1, \theta_E\right)$, as a function of θ_E and N and denoted by $G_1\left(\theta_E, N\right)$:

$$G_1\left(\theta_E, N\right) = \sum_{n-1=0}^{N-1} \binom{N-1}{n-1} \left(\theta_E\right)^{n-1} \left(1-\theta_E\right)^{N-n} \pi\left(\theta_E, n\right).$$

The entry threshold $\theta_E(N)$ implicitly solves $G_1\left(\theta_E, N\right) = 0$. Therefore,

$$\begin{aligned} &G_1\left(\theta_E(N+1), N+1\right) - G_1\left(\theta_E(N), N\right) = 0 \\ \Leftrightarrow &\left[G_1\left(\theta_E(N+1), N+1\right) - G_1\left(\theta_E(N+1), N\right)\right] \\ &+ \left[G_1\left(\theta_E(N+1), N\right) - G_1\left(\theta_E(N), N\right)\right] = 0. \end{aligned}$$

Since $\pi\left(\theta_E, n\right)$ is decreasing in n, it follows from Lemma A.2 that $G_1\left(\theta_E, N\right)$ is decreasing in N, which implies that $G_1\left(\theta_E(N+1), N+1\right) \leq G_1\left(\theta_E(N+1), N\right)$. Consequently, $G_1\left(\theta_E(N+1), N\right) \geq G_1\left(\theta_E(N), N\right)$. However, as $\pi\left(\theta_E, n\right)$ is also decreasing in θ_E, by applying Lemma A.1, we find that $G_1\left(\theta_E, N\right)$ decreases in θ_E. Therefore, it must be that $\theta_E(N+1) \leq \theta_E(N)$, which completes the proof. □

Proof of Proposition 5. The first part of the proposition directly follows from Proposition 2. In order to show that θ_E is decreasing in c, we consider $\mathbb{E}_{n-1}\left[\pi\left(\theta_E, n\right)\right]$, where $(n-1)$ follows the distribution Binomial $\left(N-1, \theta_E\right)$, as a function of θ_E and N and denote it by $G_2\left(\theta_E, c\right)$. Note that $\theta_E(c)$ implicitly solves $G_2\left(\theta_E, c\right) = 0$. Taking the total derivative of G_2 along the path of $\theta_E(c)$, we get $d\theta_E/dc = -\left(dG_2/dc\right)/\left(dG_2/d\theta_E\right)$.

Note that $\pi\left(\theta_E, c\right)$ is strictly decreasing in c, and therefore it follows from Lemma A.1 that $\partial G_2/\partial c < 0$. Further, as θ_E solves $\mathbb{E}_{n-1}\left[\pi\left(\theta_E, n\right)\right] = 0$, it must be that $\theta_E \leq \xi(n)$ at least for some n (as otherwise $\mathbb{E}_{n-1}\left[\pi\left(\theta_E, n\right)\right]$ will be independent of θ_E), which implies that $\pi\left(\theta_E, n\right)$ is strictly decreasing in θ_E for some n. Therefore, by applying Lemma A.1, we get $\partial G_2/\partial\theta_E < 0$. Hence, $d\theta_E/dc < 0$, which completes the proof. □

Proof of Proposition 6. To construct $\hat{c}\left(\theta_E\right)$, we consider the two cases separately, $\underline{c}(N) > 0$ and $\underline{c}(N) = 0$. Consider first $\underline{c}(N) > 0$. It follows from Proposition 2 that for all $c \leq \underline{c}(N)$, $\theta_E = 1$, and therefore, $\hat{c}(1) = \underline{c}(N)$. For $c > \underline{c}(N)$ and $\theta_E < 1$, θ_E and c have a one-to-one relationship satisfying (11). Therefore, for all $\theta_E < 1$, $\hat{c}\left(\theta_E\right)$ is uniquely determined by the solution of (11). Further, by Proposition 5, $\hat{c}\left(\theta_E\right)$ is strictly decreasing for $0 \leq \theta_E \leq 1$. Because $G_2\left(\theta_E, c\right)$ is continuous and differentiable in c, $\theta_E(c)$ is also continuous and differentiable in c, ensuring the continuity and differentiability of $\hat{c}\left(\theta_E\right)$ in $\theta_E \in [0,1]$. Further, because of strict monotonicity of $\hat{c}\left(\theta_E\right)$, any entry threshold θ_E in $[0,1]$ can be implemented by choosing an entry fee $c = \hat{c}\left(\theta_E\right)$. Next, consider $\underline{c}(N) = 0$, which occurs when $\xi(N) \leq 1$. We claim that if $c = 0$, then $\theta_E = 1$ is a unique solution of (11). The proof follows from two observations. Firstly, at $\theta_E = 1$, $\mathbb{E}_{n-1}\left[\pi(1,n)\right] = \pi(1,N) = 0$. Secondly, with $c = 0$, we have for all $\theta_E < 1$, $\pi\left(\theta_E, 1\right) > 0$ and $\pi\left(\theta_E, n\right) \geq 0$ for $n \geq 2$. Therefore, $\mathbb{E}_{n-1}\left[\pi\left(\theta_E, n\right)\right] > 0$ for all $\theta_E < 1$, implying that any $\theta_E < 1$ cannot be a solution of (11) if $c = 0$. For $c > 0$ and $\theta_E < 1$, θ_E and c have a one-to-one relationship satisfying (11), and therefore, $\hat{c}\left(\theta_E\right)$ is uniquely determined by the solution of (11). Further, as we have argued in the previous case, $\hat{c}\left(\theta_E\right)$ is differentiable and strictly decreasing in θ_E for all $\theta_E \in [0,1]$. Therefore, any entry threshold θ_E in $[0,1]$ can be implemented by setting $c = \hat{c}\left(\theta_E\right)$. □

Proof of Proposition 7. Total expected effort is given by (17), with further dependency on θ_E in (14) and (15). Write $\tau(n, \theta_E) = \mathbb{E}[\,T(n,m) \mid n\,]$ for this proof. Differentiating total expected effort term-by-term gives,

$$\frac{d}{d\theta_E}\mathbb{E}[T] = \sum_{n=0}^{N}\binom{N}{n}\left(\frac{d}{d\theta_E}\left[\theta_E^n(1-\theta_E)^{N-n}\right]\cdot\tau(n;\theta_E) + \theta_E^n(1-\theta_E)^{N-n}\cdot\tfrac{\partial}{\partial\theta_E}\tau(n;\theta_E)\right).$$

At $\theta_E = 1$, terms with $n \leq N-2$ vanish because they are multiplied by $(1-\theta_E)^{N-n}$ or its derivative. The only surviving probability mass function-derivative terms are for $n = N$ and $n = N-1$, giving $N\tau(N,1) - N\tau(N-1,1)$. Among the $\partial\tau/\partial\theta_E$ terms, only $n = N$ survives because $\theta_E^n(1-\theta_E)^{N-n}$ vanishes at 1 for $n \leq N-1$; for $n = N$ it equals 1, yielding the final term $\partial\tau(N,\theta_E)/\partial\theta_E|_{\theta_E=1}$. This gives the expression in the Proposition. □

Proof of Proposition 8. It follows from the discussion in Section 6.3 that $\hat{c}\left(\theta_E\right)$ is strictly positive for all $\theta_E < 1$. Therefore, the maximum value of $V_f\left(\theta_E\right)$ must be positive, and it must reach its maximum at some $\theta_E > 0$.

Let us first consider the case $\underline{c}(N) = 0$. Then, $\hat{c}(1) = \underline{c}(N) = 0$, and therefore, $V_f(1) = 0$, which implies that V_f is maximized at some interior $\theta_E \in (0,1)$, and so the designer prefers limited entry.

Next, consider $\underline{c}(N) > 0$. Therefore, $\hat{c}(1) = \underline{c}(N) = \xi(N) - 1$. We derive a sufficient condition for an interior maximum by examining the derivative of V_f as θ_E approaches 1: if the derivative is negative, then V_f must be maximized at some $0 < \theta_E < 1$. Note that

$$\lim_{\theta_E\to 1}\frac{dV_f}{d\theta_E} = N\left(\hat{c}(1) + \lim_{\theta_E\to 1}\frac{d\hat{c}\left(\theta_E\right)}{d\theta_E}\right).$$

Recall from the proof of Proposition 5 that for $\theta_E \in (0,1)$ and $c > 0$, $\hat{c}\left(\theta_E\right)$ solves

$$G_2\left(\theta_E, c\right) = \sum_{n=1}^{N}\binom{N-1}{n-1}\left(\theta_E\right)^{n-1}\left(1-\theta_E\right)^{N-n}\pi\left(\theta_E, n\right) = 0.$$

From the total differential of $dG_2 = 0$ along the path of $\hat{c}\left(\theta_E\right)$, we can derive $d\hat{c}\left(\theta_E\right)/d\theta_E = -\left(\partial G_2/\partial\theta_E\right)/\left(\partial G_2/\partial c\right)$. Further,

$$dG_2/dc = \sum_{n=1}^{N}\binom{N-1}{n-1}\left(\theta_E\right)^{n-1}\left(1-\theta_E\right)^{N-n}\frac{d\pi\left(\theta_E, n\right)}{dc} = -1,$$

which gives us $d\hat{c}\left(\theta_E\right)/d\theta_E = \left(dG_2/d\theta_E\right)$, and

$$\lim_{\theta_E\to 1}\frac{dV_f}{d\theta_E} = N\left(\xi(N) - 1 + \lim_{\theta_E\to 1}\frac{dG_2\left(\theta_E, c\right)}{d\theta_E}\right). \tag{A.5}$$

Differentiating $G_2\left(\theta_E, c\right)$ with respect to θ_E, term by term, and taking the limit as $\theta_E \to 1$, we get

$$\begin{aligned} \lim_{\theta_E\to 1}\frac{dG_2\left(\theta_E, c\right)}{d\theta_E} &= \left[(N-1)\lim_{\theta_E\to 1}\pi\left(\theta_E, N\right) + \lim_{\theta_E\to 1}\frac{d\pi\left(\theta_E, N\right)}{d\theta_E}\right] \\ &- \left[(N-1)\lim_{\theta_E\to 1}\pi\left(\theta_E, N-1\right)\right], \end{aligned} \tag{A.6}$$

where the first square-bracketed term arises from the derivative of the last term in the summation series, and the second square-bracketed term comes from the derivative of the second-to-last term of the series; Because $d\pi\left(\theta_E, n\right)/d\theta_E$ is finite for any n, it can be easily shown that the derivatives of all other terms approach zero in the limit as θ_E approaches 1.

Note that as $\xi(N) > 1$, from (10), we get $\lim_{\theta_E\to 1} d\pi\left(\theta_E, N\right)/d\theta_E = -1$. Further, as $\theta_E \to 1$, $\hat{c}\left(\theta_E\right) \to \hat{c}(1) = \underline{c}(N) = \xi(N) - 1$. Therefore,

$$\lim_{\theta_E\to 1}\pi\left(\theta_E, N\right) = \xi(N) - 1 - \hat{c}(1) = 0, \text{ and}$$

$$\lim_{\theta_E\to 1}\pi\left(\theta_E, N-1\right) = v\cdot\mathbf{1}_{\{N=2\}} + \xi(N-1) - 1 - \hat{c}(1) = v\cdot\mathbf{1}_{\{N=2\}} + \xi(N-1) - \xi(N)$$

where $\mathbf{1}_{\{N=2\}}$ is an indicator function that takes the value 1 if $N = 2$, and 0 otherwise. Replacing the limiting values in the right-hand-side of (A.6), we get

$$\lim_{\theta_E\to 1}\frac{dG_2\left(\theta_E, c\right)}{d\theta_E} = (N-1)\left(\xi(N) - \xi(N-1) - v\cdot\mathbf{1}_{\{N=2\}}\right) - 1.$$

Further, replacing the limiting value of $dG_2\left(\theta_E, c\right)/d\theta_E$ in (A.5) and using the fact that $(N-1)\,v\cdot\mathbf{1}_{\{N=2\}} = v\cdot\mathbf{1}_{\{N=2\}}$, we can express

$$\begin{aligned}\lim_{\theta_E\to 1}\frac{dV_f}{d\theta_E} &= N\left[N\xi(N)-(N-1)\,\xi(N-1)-2-v\cdot\mathbf{1}_{\{N=2\}}\right]\\ &= N\left[\xi(N-1)(1-Nq)-2-v\cdot\mathbf{1}_{\{N=2\}}\right].\end{aligned}$$

Therefore, (20) implies that $\lim_{\theta_E\to 1} dV_f/d\theta_E < 0$ and it provides a sufficient condition for having an interior maximum. □

Proof of Proposition 9. It follows from (22) that $V_{inv} = 0$ at $\theta_E = 0$, and $V_{inv} > 0$ at $\theta_E = 1$, implying that V_{inv} reaches its maximum at some $\theta_E > 0$.

Let us first consider the case $\underline{c}(N) > 0$, which occurs if $\xi(N) > 1$. In this case, $\theta_E \le \xi(n)$ for all $\theta_E \in [0,1]$ and $n \le N$, and therefore, by Proposition 1, $\theta_I = \theta_E$ and $V_{inv} = N\theta_E$, which is increasing in θ_E. Hence, the designer prefers full entry.

Next, consider $\underline{c}(N) = 0$. Then, as $\theta_E \to 1$, we have $\hat{c}\left(\theta_E\right) \to \hat{c}(1) = \underline{c}(N) = 0$. We will derive a sufficient condition for an interior maximum by examining the derivative of V_{inv} as θ_E approaches 1: if the derivative is negative, then V_{inv} must be maximized at some $0 < \theta_E < 1$. Note that

$$\lim_{\theta_E\to 1}\frac{dV_{inv}\left(\theta_E\right)}{d\theta_E} = N\lim_{\theta_E\to 1}\frac{d}{d\theta_E}\left[\sum_{n=1}^{N}\binom{N-1}{n-1}\left(\theta_E\right)^{n-1}\left(1-\theta_E\right)^{N-n}\theta_I(n)\right].$$

Differentiating $V_{inv}\left(\theta_E\right)$ with respect to θ_E, term by term, and taking the limit as $\theta_E \to 1$, we get

$$\begin{aligned}\lim_{\theta_E\to 1}\frac{dV_{inv}\left(\theta_E\right)}{d\theta_E} &= N\left[(N-1)\lim_{\theta_E\to 1}\theta_I(N)+\lim_{\theta_E\to 1}\frac{d\theta_I(N)}{d\theta_E}\right]\\ &\quad - N\left[(N-1)\lim_{\theta_E\to 1}\theta_I(N-1)\right],\end{aligned}\tag{A.7}$$

where the first square-bracketed term arises from the derivative of the last term in the summation series, and the second square-bracketed term comes from the derivative of the second-to-last term of the series; Because $d\theta_I(n)/d\theta_E$ is finite for any n, it can be easily shown that the derivatives of all other terms approach zero in the limit as θ_E approaches 1.

To find $\lim_{\theta_E\to 1} d\theta_I(N)/d\theta_E$, observe that $\theta_I(N)$ solves

$$f\left(\theta_I,\theta_E\right) := v(\alpha-1)\,q\left(1-\left(q\theta_I/\theta_E\right)\right)^{N-1}-\theta_I = 0.$$

Therefore, from the total differential of $df = 0$ along the path of $\theta_I(N)$, we can derive $d\theta_I(N)/d\theta_E = -\left(\partial f/\partial\theta_E\right)/\left(\partial f/\partial\theta_I\right)$. Further,

$$\begin{aligned}\frac{\partial f}{\partial\theta_E} &= (N-1)\,v(\alpha-1)\,q\left(1-\left(q\theta_I/\theta_E\right)\right)^{N-2}\left(\frac{q\theta_I}{\theta_E^2}\right), \text{ and}\\ \frac{\partial f}{\partial\theta_I} &= -(N-1)\,v(\alpha-1)\,q\left(1-\left(q\theta_I/\theta_E\right)\right)^{N-2}\left(\frac{q}{\theta_E}\right)-1,\end{aligned}$$

which give us

$$\lim_{\theta_E\to 1}\frac{d\theta_I(N)}{d\theta_E} = \frac{(N-1)\,v(\alpha-1)\,q^2\left(1-q\hat{\theta}\right)^{N-2}\hat{\theta}}{(N-1)\,v(\alpha-1)\,q^2\left(1-q\hat{\theta}\right)^{N-2}+1},\tag{A.8}$$

where $\hat{\theta} := \lim_{\theta_E\to 1}\theta_I(N)$. Because $f\left(\theta_I,\theta_E\right)$ is continuous in θ_I and $\theta_E > 0$, $\hat{\theta}$ satisfies $\hat{\theta} = v(\alpha-1)\,q\left(1-q\hat{\theta}\right)^{N-1}$. Therefore, (A.8) can be simplified as

$$\lim_{\theta_E\to 1}\frac{d\theta_I(N)}{d\theta_E} = \frac{(N-1)\,q\hat{\theta}^2}{(N-1)\,q\hat{\theta}+\left(1-q\hat{\theta}\right)} = \frac{(N-1)\,q\hat{\theta}^2}{(N-2)\,q\hat{\theta}+1}\in(0,1).\tag{A.9}$$

Eqs. (A.7)–(A.9) evaluate the slope of $V^{\text{inv}}(\theta_E)$ as $\theta_E \to 1$. A negative limit means that when full entry is almost achieved, nudging the fee to slightly reduce entry raises expected investment (fewer entrants but a higher success share). This is the formal counterpart to the designer's trade-off we emphasize in Section 7.

We define $\hat{\hat{\theta}} := \lim_{\theta_E\to 1}\theta_I(N-1)$, which solves $v(\alpha-1)\,q(1-q\theta)^{N-2} - \theta = 0$. It can be easily verified that $\hat{\hat{\theta}} > \hat{\theta}$.

Substituting (A.9) into (A.7) yields

$$\lim_{\theta_E\to 1}\frac{dV_{inv}\left(\theta_E\right)}{d\theta_E} = N\left[(N-1)\left(\hat{\theta}-\hat{\hat{\theta}}\right)+\frac{(N-1)\,q\hat{\theta}^2}{(N-2)\,q\hat{\theta}+1}\right].$$

Therefore, (23) implies that $\lim_{\theta_E\to 1} dV_{inv}/d\theta_E < 0$ and it provides a sufficient condition for having an interior maximum. □

Data availability

No data was used for the research described in the article.

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