Investing to get a head start in contests

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Abstract

In a contest, rivals compete for a prize by making irretrievable outlays. Commonly, researchers assume that contestants spontaneously start to compete at some designated time. In practice, however, preparation is an important part of the actual contest, where rivals may undertake actions in order to improve their chance of winning the prize. We model this as a two-player all-pay auction under complete information in which rivals can make an investment to try to get a head start before the contest is played. We provide conditions under which a pure-strategy equilibrium exists in which the player with the high prize valuation invests and the opponent does not. A pure-strategy equilibrium can also exist in which it is the player with the low prize valuation that makes the investment. The investment opportunity allows the rivals to move some of the competition to the investment stage, reducing the intensity of fighting in the contest. An important feature of our model is that investment does not necessarily preclude fighting in the contest. We show when investment leads to active participation in the contest, and when it leads a rival to simply capitulate. This is an important distinction, especially if the overall aim is to minimize contest effort, such as in armed conflicts.

Keywords: all-pay auction, pre-contest investment, endogenous head start

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1 Introduction

A contest is a situation in which rivals compete for a prize by making irretrievable outlays that may be monetary or physical in nature. Contests have been used to model a wide range of settings such as R&D and innovation competitions, political rivalry, crowd sourcing, sports contests, battles for promotion in local labor markets, and violent global conflict.[1] This paper looks at the incentives for rivals to invest in pre-contest activity that improves their chance of winning the ultimate prize, awarded in our framework through an all-pay auction with complete information.

Many years ago, Danhof (1968) noted that an important part of actual competition is influencing the ability to compete; and this was echoed by Lichtenberg (1988) in the case of public procurement. In their model of pre-contest investment, Fu and Lu (2009) pointed out a lack of literature dealing with this phenomenon. Little literature has followed, although this would seem to be an important addition to the study of contests.[2] Most researchers consider just the contest stage of competition between rivals, discounting how they may prepare themselves to achieve the largest probability of winning. Recognizing that contests do not just happen in a vacuum, we investigate how these preparations affect play in the actual contest.

A limited number of models do indeed have features that incorporate how pre-contest investment can affect the abilities of rivals to compete. Keskin and Sağlam (2019), Schaller and Skaperdas (2020) and Clark et al. (2025) assume that the effect of investment is to improve the efficiency of contest effort, while Clark and Kundu (2025) consider skill-enhancing investment prior to the contest stage. Grossmann and Dietl (2009) and Grossmann et al. (2011) look at rivals who build up human capital to be able to compete. Münster (2007) and Fu and Lu (2009) are examples of models where investment affects the cost of making contest effort, and Amegashie (2012) analyzes investment that may be destructive to rivals' efforts in the contest.

This paper looks at rivals who can invest in getting a head start that directly affects the probability that they win the contest prize. Previous literature has mostly considered a head start as being exogenously fixed, or endogenously decided by a contest designer to achieve a specific objective such as contest-effort maximization (Li and Yu, 2012; Franke et al., 2018). In a Tullock model, Arbatskaya and Mialon (2012) and Schaller and Skaperdas (2020) allow pre-contest investments that affect

[1] See Konrad (2009) and Vojnović (2015) for an overview.

[2] Konrad (2002) seems to be the first to consider investment in an all-pay auction. In his model, an incumbent could unilaterally invest to increase the size of the potential prize that could be appropriated by a future contest winner. We note the similarities between investments in our setting and entry accommodation in studies in industrial organization following the seminal piece by Fudenberg and Tirole (1984).

the probability of winning, having a similar effect as a head start.[3] In a series of all-pay auctions, Clark and Nilssen (2018) suppose that winning previous contests can give head starts in future ones through a momentum effect; the size of the momentum is exogenously fixed, however. The current paper seeks to endogenize the size and direction of the head start that a player can achieve by relating it directly to the investment made by the players. Specifically, in an all-pay auction, we allow pre-contest play in which rivals can invest in trying to get a head start; a player with a head start has a larger probability of winning than without the head start.

We find that in any pure strategy equilibrium, only one player makes a positive investment. An equilibrium exists in which the leader (with high valuation) invests while the opponent does not. When the investment has sufficiently low effect in deciding the overall contest score, there will be fighting in the contest. At a threshold level for the strength of investment, the leader builds up such a large head start that the rival does not contest the prize, resulting in no fighting. An important finding is the existence of an equilibrium in which the laggard (the player with low valuation) is the investing player, whilst the opponent does not invest. Depending on the strength of the investment, this equilibrium may involve fighting or not at the contest stage. Delineating cases in which fighting occurs is important for applications in which this activity is seen as negative for society (such as armed conflicts) and should be minimized.

Our all-pay auction presents strict incentives to invest since one of the two contestants will always expect a payoff of zero. This is noted previously by Hirata (2014) in a two-period model of an all-pay auction where players can bid in each of two periods and the winner has the higher sum of bids. He shows that any equilibrium is such that the player with the lower prize valuation will not bid in the first period. The first-period bid of the high-value player thus becomes his head start in the second period; the net expected payoff to the high-value player is the difference in valuations minus the first round bid. This is lower than the expected payoff in a static all-pay auction with complete information, so that the contest designer can use this mechanism to increase expected revenue.

The main case considered by Hirata (2014) has a continuum of pure-strategy equilibria in which the high-value player makes a bid in the first period but this is not sufficiently high to deter fighting in the second period of the contest. To remove the indeterminacy of equilibrium, Hirata (2014) then considers a perturbed model in which bids in the first period are weighted more than those made later. Then,

[3] Baik and Lee (2000) and Schmitt et al. (2004) entertain the possibility that effort spillovers from the first rounds of a multi-period contest can give a head start in a Tullock model.

two pure-strategy equilibria emerge and may coexist: one in which the high-value player only makes a large investment and one in which the low-value player is the sole investor. In both cases, the first-period bid (i.e., the head start) is so large that there is no fighting in period two.

There are several important distinctions between our model and that of Hirata (2014). Rivals in our model undertake different activities in the two periods, and this is captured by the fact that the pre-contest investment has a convex cost function, while there is a constant unit cost of effort in the contest.[4] Hence, we can consider applications in which rivals *prepare* to contest a reward, rather than portioning out their effort over several periods in the same contest.[5] Furthermore, our model admits equilibria in which the high or low-value player is the sole investor, and in which there will actually be fighting at the contest stage. In contrast to Hirata (2014), rivals in our framework can invest to increase their probability of winning, but not always a sufficient amount to deter fighting in the contest. In addition, we discuss how to resolve the multiplicity of pure strategy equilibria, and how this affects fighting at the contest stage. We demonstrate that the equilibria in Hirata (2014) obtain when the cost of investing is linear, and that the equilibria with contest effort are an additional possibility for any strictly convex cost of investment function. Finally, we change the game to allow for sequential investments.

Our model gives equilibria in which only one player invests, and this may be viewed as objectionable for explaining how rivals prepare for a contest. To some degree, all competitors prepare for the upcoming battle. There are, however, several examples in which it would seem that a single competitor has made preparations that give a sizable advantage in a contest. In sports, the Norwegian 400-meter hurdler Karsten Warholm and his coaching team developed a new running shoe ("The Berserker") with equipment company Puma; these shoes are widely thought to give an advantage in competition.[6] Football coach Rinus Michels is widely credited with developing the Total Football system for the modern era, with which the Dutch national team became runners-up in the 1974 and 1978 World Cups. In turn, this system has been further developed ("Tiki-taka") to form a foundation for the success of Barcelona and the Spanish national football team.[7] Whilst all competitors are preparing for competition in these examples, one is making an ex-

[4] Arbatskaya and Mialon (2012) and Schaller and Skaperdas (2020) consider Tullock models in which contestants undertake different activities at each stage.

[5] Hirata (2014) extends his basic model to a general cost function for total effort. As he notes, the division of effort in each period does not matter for total cost, and this is not realistic in many applications.

[6] See The Guardian, 29th May 2024.

[7] See Wikipedia on Total football.

traordinary investment that may improve the odds of winning. Not all activities may be considered ethical or legal, such as the use of performance-enhancing drugs in sport.[8]

A related phenomenon in business is that of market shaping by actors seeking to transform existing practices or create new products.[9] Examples here are the TV channel MTV, Apple's iPhone and the development of Amazon Marketplace; the actors here invested in developing a new product or practice that allowed them to influence the coming competition for viewers and customers. Day and Schoemaker (2016) document the case of the pharmaceutical company Novartis which in 2012 equipped its sales representatives with mobile units capable of two-way video communication in real time with prescribing physicians. This changed the ruling "Share of Voice" model in which sales representatives typically engaged (typically unengaged) potential clients with a scripted monologue. This effectively changed the contest among the pharmaceutical companies, gaining Novartis an advantage. These examples indicate that a single player can and will invest in an activity that gives a head start on the rivals; of course, over time the rivals can catch up this initial lead, and then the process of investment before a new contest repeats. Our analysis presents a snapshot of the combined phases of one of these competitions.

The paper is organized as follows. Section 2 introduces the model, and Section 3 analyzes the second-period contest. The incentive that each player has to invest is introduced in Section 4, and the different types of equilibrium and an example is presented in Section 5. Section 6 discusses the results; specifically, we consider how to resolve the multiplicity of pure-strategy equilibria, and we explore the existence of mixed-strategy equilibrium. In Section 7, we change the model to allow sequential investment in order to test the robustness of our model predictions. Section 8 concludes. All proofs are in the Appendix.

2 The model

Consider two risk neutral players involved in a two-period game. In period 1, each of them makes an up-front investment in order to gain a head-start in the upcoming contest. Let $k_i \geq 0$ be the investment of player i in period 1, with strictly increasing and convex investment cost $c(k_i)$, where $c(0) = 0; c'(0) = 0; c'(k_i) > 0, k_i > 0;$ and $c''(k_i) > 0$. In period 2, the players play an all-pay auction with head

[8] See Berentsen (2002) and Haugen (2004) for early game-theoretic analyses of doping in sports.
[9] See Kaartemo and Nyström (2021) and the references therein.

starts. Player 1 has a higher valuation of the contest prize: $V_1 > V_2 > 0$.[10]

The players' win probabilities in the period-2 contest depend on their efforts in period 2 as well as their investments in period 1. In particular, player $i \in \{1, 2\}$ has a score in this contest, denoted S_i, consisting of his gross head start and his effort:

$$S_i = h_i + x_i = sk_i + x_i.$$

Here, x_i is the player's effort in the period-2 contest, while $h_i = sk_i$ is the player's gross head start. The parameter $s > 0$ measures how effective an investment is at creating a head start.

A player's expected (gross) payoff from taking part in the period-2 contest is

$$EU_i \ = \rho_i V_i - x_i,$$

where ρ_i is the player's probability of winning the contest. The period-2 contest is an all-pay auction. The probability of player $i \neq j$ winning the contest is:

$$\rho_i(S_1, S_2) = \begin{cases} 1, & \text{if } S_i > S_j; \\ \frac{1}{2}, & \text{if } S_i = S_j; \text{and} \\ 0, & \text{if } S_i < S_j. \end{cases}$$

The probability of player j winning is $1 - \rho_i(S_1, S_2)$.

A player's expected net payoff in this game, $EW_i(k_i, k_j)$, is the expected gross payoff from taking part in the contest, net of any investment costs:

$$EW_i(k_i, k_j) = EU_i - c(k_i).$$

Define

$$\Delta_1 \ = h_1 - h_2 = s\,(k_1 - k_2)$$

as player 1's (net) head start. Similarly, we have $\Delta_2 = -\Delta_1$ for player 2. We are looking for subgame-perfect Nash equilibria of this two-period game.

3 The period-2 contest

We start out with analysing the period-2 game, where Δ_1 is fixed. This game has a unique equilibrium, which is in mixed strategies – except when the head start is

[10] The case of symmetric valuations is a straightforward special case of the results we present. Our focus is on how investment can enhance or reduce an initial asymmetry.

large. The following result is based on Clark and Riis (1995); see also Lemma 1 in Clark and Nilssen (2021). Recall that $\Delta_1 < 0$ means that $\Delta_2 = -\Delta_1 > 0$ so that player 2 has the head start.

Lemma 1.

1. *If $\Delta_1 \geq V_2$, then the unique equilibrium in the period-2 contest is in pure strategies, $x_1 = x_2 = 0$, with expected payoffs $EU_1 = V_1$ and $EU_2 = 0$.*

2. *If $0 \leq \Delta_1 < V_2$, then the unique mixed-strategy equilibrium in the period-2 contest has expected efforts and payoffs given by*

$$Ex_1 = \frac{1}{2V_2}\left(V_2 - \Delta_1\right)^2; Ex_2 = \frac{1}{2V_1}\left(V_2^2 - \Delta_1^2\right);$$

$$EU_1 = V_1 - V_2 + \Delta_1; EU_2 = 0.$$

3. *If $-(V_1 - V_2) < \Delta_1 < 0$, then the unique mixed-strategy equilibrium in the period-2 contest has expected efforts and payoffs given by*

$$Ex_1 = \frac{V_2}{2} - \Delta_1; Ex_2 = \frac{V_2^2}{2V_1};$$

$$EU_1 = V_1 - V_2 + \Delta_1 \geq 0; EU_2 = 0.$$

4. *If $-V_1 < \Delta_1 \leq -(V_1 - V_2)$, then the unique mixed-strategy equilibrium in the period-2 contest has expected efforts and payoffs given by*

$$Ex_1 = \frac{V_1(V_1 + \Delta_1)}{2V_2}; Ex_2 = \frac{(V_1 + \Delta_1)^2}{2V_1};$$

$$EU_1 = 0; EU_2 = V_2 - V_1 - \Delta_1 \geq 0.$$

5. *If $\Delta_1 \leq -V_1$, then the unique pure strategy equilibrium in the period-2 contest has $x_1 = x_2 = 0$, with expected payoffs $EU_1 = 0$ and $EU_2 = V_2$.*

Lemma 1 delineates five different cases according to the players' prize valuations and head starts. Note that two of them, cases 1 and 5, involve no fighting: the difference between the two players is so large that it does not pay for the lower-placed player to expend effort. In the three other cases, there is still a difference among the players, but it is sufficiently small that it pays for the lower-placed player to put up a fight, even though he may not get any net benefit from his effort in equilibrium. This is key for disentangling the effects that investment has on contest efforts in equilibrium.

Since the contest happens in the second-period, it is the expected payoffs that are important for calculating best responses at the investment stage. We shall see that – in terms of best responses – case 3 of Lemma 1 will not occur in an equilibrium with investment, since it involves player 2 making a positive investment but still having an expected payoff of zero in the contest. Note also that cases 1 and 2 converge in terms of expected payoff when $\Delta_1 \to V_2$, even though the actual equilibrium strategies employed are different in these cases. The same applies to cases 4 and 5 when $\Delta_1 \to -V_1$. Hence, the best responses that we derive for the investment stage may not be unique.

4 Investment incentives

Lemma 1 shows that player 1, who has the higher valuation of the prize, will come positively out of the contest in expectation, except when the other player has built up a net head start that at least matches the difference in their valuations. Moreover, only one of the players expects a positive payoff in the contest, implying that only one player has an incentive to invest at the first stage.[11] This is stated as:

Lemma 2. *There is no pure-strategy equilibrium in which both players simultaneously make non-zero investments.*

A pure-strategy equilibrium can thus be derived by assuming that player i does not invest, working out the optimal investment of player j, and then confirming that no investment is indeed a best response for player i. We first consider the incentives for the high-valuation player to invest before turning to the player with the lower valuation.

Consider the incentive of the high valuation player 1 to invest given $k_2 = 0$, depicted in Figure 1, which plots the net expected payoff against the investment level of player 1.

By investing, player 1 has the possibility of forcing case 1 or case 2 in Lemma 1 at the contest stage. The red curve in Figure 1 is the expected net payoff if case 1 obtains and the head start is so large ($k_1 \geq \frac{V_2}{s}$) that player 1 wins the contest without opposition. Once the player's investment reaches the level at which the rival gives up, further investment just represents a cost, and hence this payoff function slopes downwards. The black curve represents case 2 in which player 1 has a head

[11] This is in line with Münster (2007) and Hirata (2014).

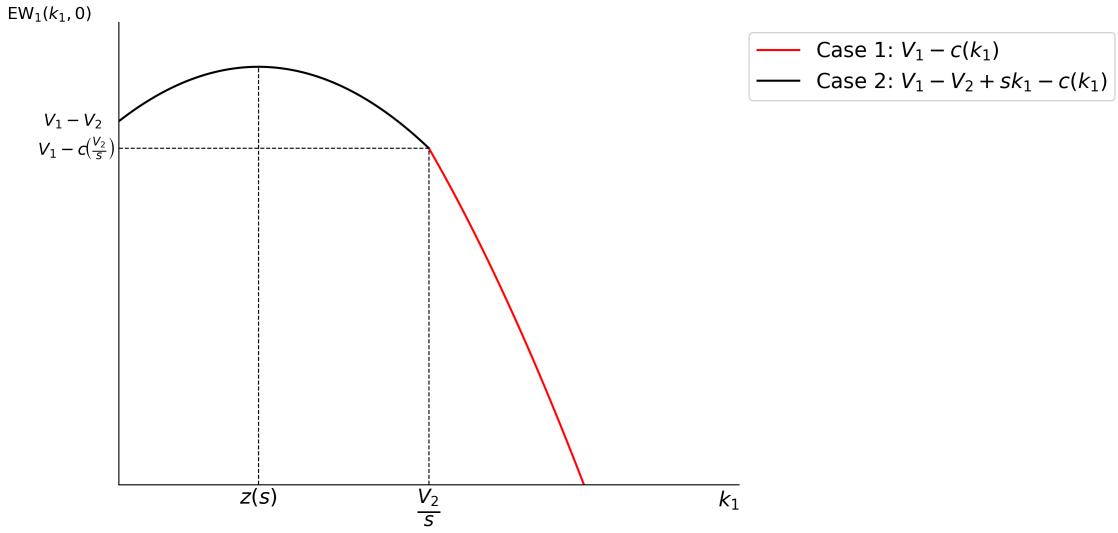

Figure 1: Player 1's investment incentive

start but the rival puts up a fight in the contest. Note that these payoff functions meet at $k_1 = \frac{V_2}{s}$, and the payoff functions are continuous at this point.

From Lemma 1, case 2 obtains to the left of the crossing point, and case 1 to the right. This means that the choice to be made by player 1 is traced out by the black line for low values of k_1 and the red one for higher. The payoff in case 2 is maximized for $c'(k_1) = s$; denote this level of investment by $z(s) := c'^{-1}(s)$. It is clear that $k_1 = z(s)$ maximizes player 1's net expected payoff in case 2, making it a candidate for the equilibrium investment by this player. Any investment above this causes the expected profit to fall, first along the black curve, and then along the red one for $k_1 \geq \frac{V_2}{s}$. In the continuation game, there will be fighting at the contest stage following investment $k_1 = z(s)$.

Formally, the optimal choice of investment in case 2 is defined as

$$z(s) := \arg\max_{k \geq 0} \; sk - c(k),$$

and the maximum value function as

$$M(s) := sz(s) - c(z(s)).$$

The function $z(s)$ is strictly increasing in s, and can be concave, convex or linear, depending on the third derivative of the cost function. Our assumption that $c'(0) = 0$ ensures that $z(s) > 0$ and $M(s) > 0$. It follows from the envelope theorem that $M'(s) > 0$.

Figure 1 is drawn for the case $\frac{V_2}{s} > z(s)$. When $\frac{V_2}{s} < z(s)$, the payoff functions cross to the left of $k_1 = z(s)$, making $\frac{V_2}{s}$ a candidate for player 1's investment level in equilibrium.[12] The continuation game from this point will not involve fighting at the contest stage.

We can establish the link between our model and that of Hirata (2014) by looking at the incentive of player 1 when the investment cost is linear, as in Hirata (2014), illustrated in Figure 2.[13]

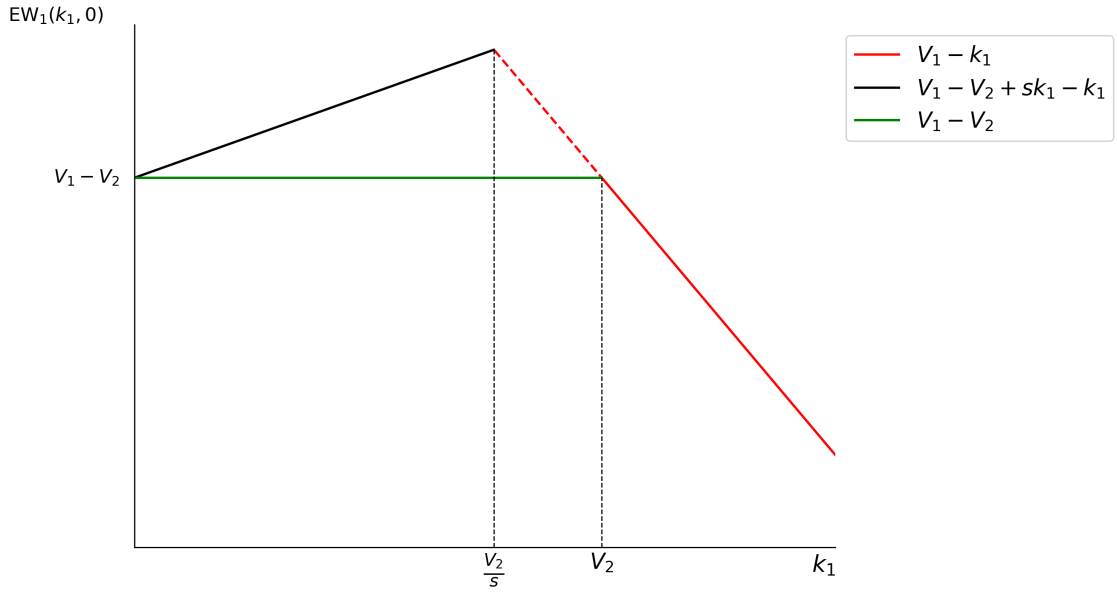

Figure 2: Investment incentives in Hirata (2014)

The red line is the linear payoff in the case with no fighting in the continuation. Two lines are drawn for the case in which there is fighting following investment, and these correspond to different versions of the model in Hirata (2014). In his basic model, $s = 1$, and the payoff is constant at $V_1 - V_2$ up to $k_1 = V_2$ (green line). In Figure 2, player 1 is indifferent between $k_1 \in [0, V_2]$, and will not invest above this, since the payoff then falls, following the red line. This illustrates Proposition 2 in Hirata (2014), in which there is a continuum of equilibria, each of which is characterized by an investment level in this range. Hirata (2014) considers a perturbed version of the basic model in his Section 3.2 in which $s > 1$, resulting in the black payoff line up to $k_1 = \frac{V_2}{s}$ and the red lines afterwards in Figure 2. Following

[12] We do not illustrate this for player 1. However, Figure 3 depicts the corresponding case for player 2.

[13] Hirata (2014) formally considers spreading effort over two periods of a contest. Our discussion of Figure 2 translates his context to ours with investment in period 1 and contest effort in period 2. His model does not strictly speaking fit our assumptions, since we assume a strictly convex cost function.

our previous logic, this makes $k_1 = \frac{V_2}{s}$ the only candidate for equilibrium, since increasing above this sends 1's payoff downwards along the dotted red line. This result appears in Proposition 4 in Hirata (2014). In both versions of Hirata (2014), the high-valuation player invests just enough to preclude fighting in the contest. In our model, a general convex investment cost leads to a concave payoff function for cases 2 and 4 in Lemma 1, permitting an investment level that can lead to effort being made in the contest.

Consider now the incentive of player 2 to invest. With a lower prize valuation than player 1, player 2 is the underdog but can attempt to overcome his drawback by overtaking the rival at the investment stage. He does this by forcing the contest into case 4 or case 5 of Lemma 1.

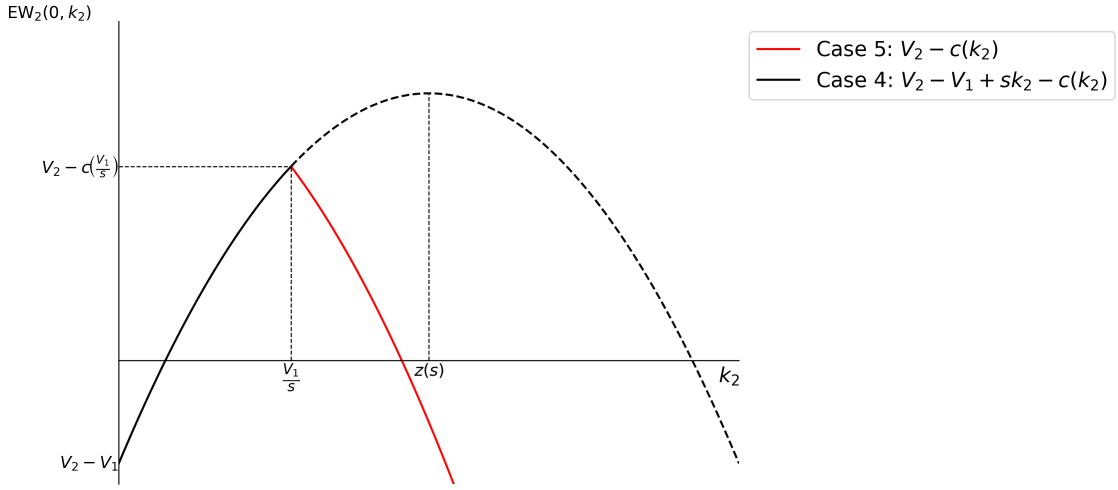

Figure 3: Player 2's investment incentive

Figure 3 depicts the net expected payoff functions of player 2 given $k_1 = 0$. The red curve is the net expected payoff of player 2 when he gets such a large head start that case 5 obtains and he wins the contest without a fight. The black curve arises if there is effort at the contest stage (case 4), indicating that a certain amount of investment is necessary in order to achieve a positive expected payoff in this case. Figure 3 is drawn for $\frac{V_1}{s} < z(s)$.[14] Note that the payoff functions meet at $k_2 = \frac{V_1}{s}$, and that case 4 (whole black line) is valid to the left of this point, and case 5 (red line) to the right. The investment choice that maximizes player 2's expected net payoff is hence $k_2 = \frac{V_1}{s}$ in this case, making it a candidate for equilibrium. In the continuation game, player 2 wins the contest without a fight. The top point of the expected payoff in case 4 ($k_2 = z(s)$) is unattainable since investment above

[14]The dashed black line is included to indicate the position of $z(s)$ but this is not an attainable payoff here.

$k_2 = \frac{V_1}{s}$ forces the game into case 5. If $\frac{V_1}{s} > z(s)$, then the crossing point of the payoff functions is to the right of the maximum payoff for case 4, and $k_2 = z(s)$, making this maximum in case 4 attainable and optimal. In the continuation game, there will then be fighting at the contest stage.

Lemma 3 characterizes each player's best response as $\tilde{k}_i(k_j)$. Note that the best responses are not necessarily unique; recall the discussion of Lemma 1.

Lemma 3. *Fix $k_2 \geq 0$. The best response of player 1 is:*

1A: *If $z(s) \geq c^{-1}(V_1)$, then*

$$\tilde{k}_1(k_2) = \begin{cases} k_2 + \frac{V_2}{s}, & \text{for } k_2 \leq c^{-1}(V_1) - \frac{V_2}{s}, \\ 0, & \text{for } k_2 \geq c^{-1}(V_1) - \frac{V_2}{s}. \end{cases}$$

1B: *If $z(s) \leq c^{-1}(V_1)$, then*

$$\tilde{k}_1(k_2) = \begin{cases} k_2 + \frac{V_2}{s}, & \text{for } k_2 \leq z(s) - \frac{V_2}{s}, \\ z(s), & \text{for } k_2 \in \left[z(s) - \frac{V_2}{s},\ z(s) - \frac{c\left(z(s)\right)}{s} + \frac{V_1 - V_2}{s}\right], \\ 0, & \text{for } k_2 \geq z(s) - \frac{c\left(z(s)\right)}{s} + \frac{V_1 - V_2}{s}. \end{cases}$$

Fix $k_1 \geq 0$. The best response of player 2 is:

2A: *If $z(s) \leq c^{-1}(V_2)$, then*

$$\tilde{k}_2(k_1) = \begin{cases} k_1 + \frac{V_1}{s}, & \text{for } k_1 \leq z(s) - \frac{V_1}{s}, \\ z(s), & \text{for } k_1 \in \left[z(s) - \frac{V_1}{s},\ z(s) - \frac{c\left(z(s)\right)}{s} - \frac{V_1 - V_2}{s}\right], \\ 0, & \text{for } k_1 \geq z(s) - \frac{c\left(z(s)\right)}{s} - \frac{V_1 - V_2}{s}. \end{cases}$$

2B: *If $z(s) \geq c^{-1}(V_2)$, then*

$$\tilde{k}_2(k_1) = \begin{cases} k_1 + \frac{V_1}{s}, & \text{for } k_1 \leq c^{-1}(V_2) - \frac{V_1}{s}, \\ 0, & \text{for } k_1 \geq c^{-1}(V_2) - \frac{V_1}{s}. \end{cases}$$

5 Equilibrium

We now derive the subgame perfect pure-strategy equilibria of the game. From Lemma 2 and Lemma 3, we know that only one player will invest in equilibrium,

and if player i is the only investor, he will choose between the investment levels $k_i \in \left\{z(s), \frac{V_j}{s}\right\}$. Furthermore, the discussion of investment incentives, illustrated in Figures 1 and 3, suggests that player i will choose the lower of the two values, $k_i = \min\left\{z(s), \frac{V_j}{s}\right\}$. Since $sz(s)$ is increasing in s, there exists a unique threshold $\hat{s}_i$ such that $z(s) < \frac{V_j}{s}$ if and only if $s < \hat{s}_i$. Specifically,

$$\hat{s}_i = \left\{s > 0 : sz(s) = V_j, j \neq i, \right\}, i, j \in \{1, 2\}.$$

Below we first present equilibria in which the strong player 1 makes the investment, reinforcing his position as the favorite. Subsequently, we look at equilibria that involve the weaker player 2 investing enough so as to overcome his initial disadvantage.

5.1 Reinforcement

In a reinforcement equilibrium, player 1's optimal investment level is $k_1 = \frac{V_2}{s}$ for $s > \hat{s}_1$. From Lemma 3, we know that player 2 can respond with a positive level of investment, specifically, by choosing $k_2 = \min\left\{z(s), \frac{V_1+V_2}{s}\right\}$, provided this yields a non-negative payoff. As s increases, player 2's payoff from investing also increases, eventually making it profitable to deviate from $k_2 = 0$. At that point, the reinforcement equilibrium ceases to exist. To sustain the equilibrium, the range of s needs to be limited.[15] To this end, we define

$$\bar{s}_1 := \begin{cases} \dfrac{V_1 + V_2}{c^{-1}(V_2)} & \text{if } z\left(\dfrac{V_1 + V_2}{c^{-1}(V_2)}\right) > c^{-1}(V_2) \\ M^{-1}(V_1) & \text{if } z\left(\dfrac{V_1 + V_2}{c^{-1}(V_2)}\right) \leq c^{-1}(V_2) \end{cases}. \tag{1}$$

The following proposition shows that $\bar{s}_1$ constitutes an upper bound on s for the existence of a reinforcement equilibrium. The two possible values of $\bar{s}_1$ are derived from the conditions under which player 2 obtains a negative payoff when choosing from candidates for a best response, $\frac{V_1+V_2}{s}$ and $z(s)$, respectively.

Proposition 1. *For $0 < s \leq \bar{s}_1$, a pure-strategy equilibrium of this game exists in which the high-value player 1 invests in period 1, while player 2 invests nothing and earns nothing.*

1. *For $0 < s < \hat{s}_1$, player 1 invests $k_1^* = z(s)$ and earns an expected net payoff of $V_1 - V_2 + M(s) > 0$. Expected efforts in the period-2 contest are positive*

[15] In his comparable model, Hirata (2014) also limits the parameter space. We discuss equilibria outside of our bounds in Section 6.2.

and given by

$$E(x_1) = \frac{1}{2V_2}\big(V_2 - sz(s)\big)^2; \quad E(x_2) = \frac{1}{2V_1}\big(V_2^2 - [sz(s)]^2\big).$$

2. *For $\hat{s}_1 \leq s \leq \bar{s}_1$, player 1 invests $k_1^* = \frac{V_2}{s}$ and earns an expected net payoff of $V_1 - c\big(\frac{V_2}{s}\big) > V_1 - V_2 > 0$, and there is no effort in the period-2 contest.*

In this equilibrium, player 1 builds upon the initial asymmetry in valuations by using the investment rule $k_1^* = \min\big\{z(s), \frac{V_2}{s}\big\}$. For low values of s in part 1, investment is $z(s)$. The fact that the investment has a small effect on the head start means that player 1 does not build up a sufficient lead to preclude fighting at the contest stage. As s increases, investment increases, giving a double boost to the size of the head start. The combined effect is still too small to dissuade player 2 from fighting in the contest as long as $0 < s < \hat{s}_1$. Further increasing s makes the leader invest just that amount that makes the head start exactly V_2, so that the laggard gives up without a fight at the contest stage. Note that the expected net payoff with optimal investment lies above that in the standard asymmetric all-pay auction without investments, in which the high-valuation player expects $V_1 - V_2$. Also note that, since $\bar{s}_1 > 0$, a reinforcement equilibrium always exists over a positive range of values of s.

5.2 Preemption by the underdog

Here, we investigate the possibility that the weaker player 2 can invest to overturn the initial disadvantage and expect a positive payoff from the game. For this to be feasible, the value of s must be large enough that an investment gives a sufficient head start. On the other hand, s cannot be too large since player 1 can then respond with a positive level of investment, specifically by choosing $k_1 = \min\{z(s), k_2 + \frac{V_2}{s}\}$, as this strategy yields a positive payoff for high values of s. To sustain the preemptive equilibrium, it is therefore necessary to restrict the range of s. To this end, we define

$$\underline{s}_2 = \min\{s > 0 : M(s) = V_1 - V_2 \text{ and } c(z(s)) = V_1 - V_2\}, \tag{2}$$

and

$$\bar{s}_2 := \begin{cases} \dfrac{V_1+V_2}{c^{-1}(V_1)} & \text{if } z\left(\dfrac{V_1+V_2}{c^{-1}(V_1)}\right) > c^{-1}(V_1) \\ M^{-1}(V_2) & \text{if } z\left(\dfrac{V_1+V_2}{c^{-1}(V_1)}\right) \leq c^{-1}(V_1) \end{cases}. \tag{3}$$

Proposition 2 describes the preemptive equilibrium.

Proposition 2. *For* $\min\{\underline{s}_2, \hat{s}_2\} \leq s \leq \bar{s}_2$*, a pure strategy equilibrium of this game exists in which the low-value player 2 invests in period 1, while player 1 invests nothing and earns nothing.*

1. *For* $\underline{s}_2 \leq s < \hat{s}_2$*, player 2 invests* $z(s)$ *and earns an expected net payoff of* $M(s) - (V_1 - V_2) > 0$*. Expected effort in the period-2 contest is positive and given by*

$$E(x_1) = \frac{V_1}{2V_2}\big(V_1 - sz(s)\big), \quad E(x_2) = \frac{1}{2V_1}\big(V_1 - sz(s)\big)^2.$$

2. *For* $\hat{s}_2 \leq s \leq \bar{s}_2$*, player 2 invests* $\frac{V_1}{s}$ *and earns an expected net payoff of* $V_2 - c\big(\frac{V_1}{s}\big) > 0$*, and there is no effort in the period-2 contest in period 2.*

As noted in Proposition 2, a preemptive equilibrium exists if and only if $\min\{\underline{s}_2, \hat{s}_2\} \leq \bar{s}_2$. We show that the existence condition holds if and only if $M(\hat{s}_2) \leq V_2$. Furthermore, we find that $\bar{s}_2 < \bar{s}_1$, which implies that, whenever a preemptive equilibrium exists for some value of s, a reinforcement equilibrium also exists. The following proposition documents the above observations.

Proposition 3. *The lagging player can invest in equilibrium if and only if* $M(\hat{s}_2) \leq V_2$*. Furthermore, if a preemptive equilibrium exists for some value of* s*, a reinforcement equilibrium also exists.*

Since $M(\hat{s}_2)$ increases with V_1, the existence condition fails to hold if V_1 becomes too large relative to V_2. Hence, the difference between the players' valuations must be sufficiently small to sustain a preemptive equilibrium. The intuition is straightforward: for such an equilibrium to exist, player 1 must have no incentive to deviate from $k_1 = 0$. His temptation to reinforce his initial advantage—thereby pushing the contest toward case 1 or case 2 in Lemma 1—is largely driven by the difference between the players' valuations. This difference must remain small to dissuade deviation.

Both players have the same cost function for investment, and player 1 has an advantage in the all-pay auction *ex ante* by virtue of having the largest valuation.

This means that player 1 has more to gain from winning the contest and can make a lower investment than 2 to force a contest with no effort. For any given s, if player 2 finds it worthwhile to invest so as to preclude fighting in the contest, then player 1 will also find this worthwhile. If it is profitable for player 2 to invest $z(s)$, then player 1 will also find it profitable, since he has the same incentives at the margin as the rival when they are on the concave portion of the expected payoff function. Hence, for any value of s at which player 2 will want to be the lone investor, then player 1 will also be interested in being the sole investor. If there is a value of s for which a preemptive equilibrium exists, then there will also be a reinforcement equilibrium for that same s. Since player 2 must first catch up the advantage of the rival, there are some low values of s for which player 1 will find investment profitable but not player 2. When s is sufficiently large, player 1 will not find it optimal to set zero investment and the preemptive equilibrium will not exist; the reinforcement equilibrium may still exist, however.

In the equilibrium with preemptive investment, the low-value player catches up and overtakes the rival. Post investment, player 2 has an effective valuation before the contest comprising of his head start plus his valuation: $min\{sz(s), V_1\}+V_2 > V_1$. This inequality follows since player 2 has a positive expected payoff at the contest stage. Suppose that there is a further player with valuation V_3 such that $V_1 > V_3 > V_2$, so that player 2 still has the lowest value. The preemptive equilibrium in which player 2 invests will still be an equilibrium in this case. The intuition is straightforward: player 2 makes an investment that deters the strongest player from investing, so it will also deter investment by the intermediate player, i.e. $min\{sz(s), V_1\} + V_2 > V_1 > V_3$.[16]

5.3 Equilibrium investments – an illustrative example

Two types of equilibrium with investment occur in our model. The one that was known from Hirata (2014) involves player i investing $k_i = \frac{V_j}{s}$ in order to force the opponent to give up on fighting in the contest. Indeed our results encompass Proposition 4 in Hirata (2014). He finds that for a given parameter range for s, the strong player makes this investment; this equilibrium is unique unless s falls in an intermediate range in which another equilibrium exists where the weak player invests enough to preclude fighting in the contest. This is exactly the content of parts 2 of our Propositions 1 and 2.

The alternative investment strategy that emerges as equilibrium behavior in our

[16]There may of course be other equilibria when there are more than two players.

model is important since it leads to fighting at the contest stage; hence, we capture situations in which rivals prepare for a contest and then actually exert effort at that stage. As we have discussed, this behavior is due to the assumption of increasing, strictly convex cost. The parameter s, which measures the effect of investment on the head start, can then be used to delineate investment behaviors in equilibrium. When investment has a sufficiently strong effect on the head start, the preclusive equilibria arise and there is no effort at the contest stage. Lower investment efficiency admits equilibria with some investment, but the cost of investment increases too quickly to make it an option for players to force the rival out of the contest. Both players use the same investment schedule $z(s)$ in this type of equilibrium, in which the investment level is strictly increasing in s.

We illustrate the possibilities that arise for the investment schedules by considering a power cost function: $c(k) = \frac{k^\gamma}{\gamma}$, where $\gamma > 1$. In this case, $z(s) = s^{\frac{1}{\gamma-1}}$, which is strictly increasing in s, and is convex for $\gamma \in (1,2)$, linear for $\gamma = 2$, and concave for $\gamma > 2$. It is straightforward to calculate:

$$\underline{s}_2 = \left[\gamma\left(V_1 - V_2\right)\right]^{\frac{\gamma-1}{\gamma}}$$

$$\overline{s}_2 = \begin{cases} \dfrac{V_1+V_2}{(\gamma V_1)^{1/\gamma}}, & \text{if } V_2 > (\gamma-1)V_1, \\ \left[\frac{\gamma}{\gamma-1}V_2\right]^{\frac{\gamma-1}{\gamma}}, & \text{otherwise;} \end{cases}$$

$$\overline{s}_1 = \begin{cases} \dfrac{V_1+V_2}{(\gamma V_2)^{1/\gamma}}, & \text{if } V_1 > (\gamma-1)V_2, \\ \left[\frac{\gamma}{\gamma-1}V_1\right]^{\frac{\gamma-1}{\gamma}}, & \text{otherwise;} \end{cases}$$

$$\hat{s}_1 = V_2^{\frac{\gamma-1}{\gamma}}, \qquad \hat{s}_2 = V_1^{\frac{\gamma-1}{\gamma}}.$$

Note that $\underline{s}_2$ is defined uniquely here since, for the power cost function considered,

$$M(s) = \frac{\gamma-1}{\gamma}s^{\frac{\gamma}{\gamma-1}} > c(z(s)) = \frac{s^{\frac{\gamma}{\gamma-1}}}{\gamma} = V_1 - V_2,$$

from (2).

For the preemptive equilibrium to exist, it must be the case that $\overline{s}_2 > \underline{s}_2$, which implies that V_1 cannot be too large in relation to V_2. If the initial asymmetry is too large, the lagging player 2 cannot catch up by making a pre-contest investment.[17]

[17]Specifically, if $\overline{s}_2$ is given by the top line (requiring $2 > \gamma > 1$), the preemptive equilibrium exists for $\frac{1}{\gamma-1} > \frac{V_1}{V_2} > 1$. If the bottom line gives $\overline{s}_2$, then preemptive equilibrium requires $\frac{\gamma}{\gamma-1} > \frac{V_1}{V_2} > max\{1, \frac{1}{\gamma-1}\}$.

Figures 4 and 5 depict two possible cases in which the reinforcement and preemptive equilibria can co-exist. The figures depict how investment depends upon its efficiency as measured by s. The black investment line is valid when player 1 invests (reinforcement), and the red lines are investment by player 2 in the preemptive equilibrium. In these figures, we fix $V_2 = 1$ and $\gamma = 3$ and vary the value of V_1 across the figures.

When there is relatively little difference in the valuations of the two players, the investment schedules of the leader and the laggard can have an overlapping portion; see Figure 4, where $V_1 = 1.2$. For low values of s, only the leader invests, using $k_1 = z(s)$. When $s > \underline{s}_2$, an equilibrium also exists in which $k_2 = z(s)$. When $s = \hat{s}_1$, player 1 finds it profitable to switch investment strategy, now using $k_1 = \frac{V_2}{s}$, and this discourages the rival from making effort in the contest. A similar pattern arises for player 2, but for a higher value of s since the investment must first neutralize the lead of player 1 and then give a head start. Eventually, investment becomes so efficient, that the laggard also invests in order to preclude fighting in the contest. When $s > \bar{s}_2$, the preemptive equilibrium breaks down. Investment is so effective that it is no longer a best response for player 1 to set $k_1 = 0$. Note that when the two pure strategy equilibria co-exist, investment is weakly higher in the preemptive one for a common value of s.

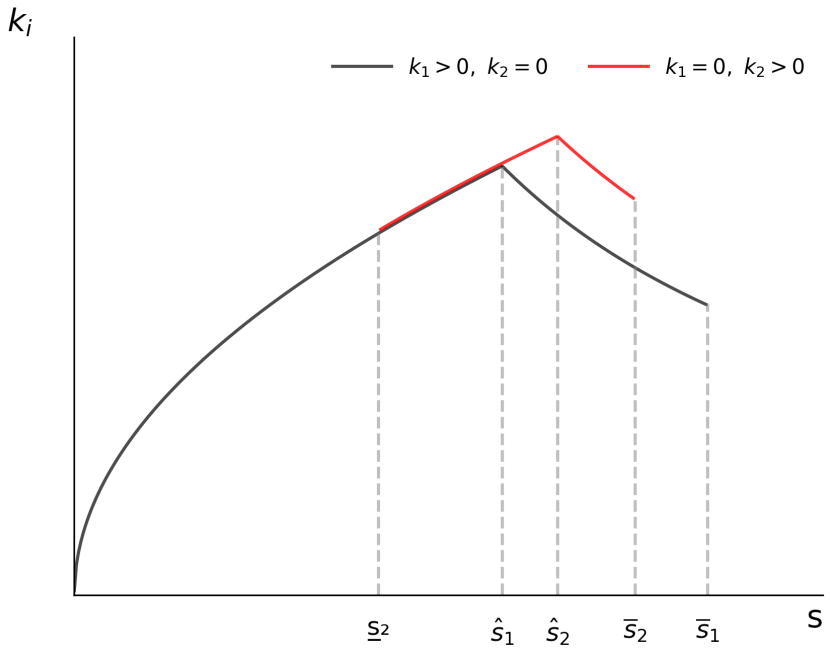

Figure 4: Investment schedules: $V_1 = 1.2, V_2 = 1, \gamma = 3$

Figure 5 depicts a case in which the value of player 1 is higher: $V_1 = 1.4$. Now the investments schedules of the players do not overlap, but both follow a strategy in

which they first set $k_i = z(s)$ and for higher s switch to $k_i = \frac{V_j}{s}$. Increasing the value of V_1 means that the laggard has more ground to make up before an investment translates into a head start in the contest. Hence, player 2 does not invest until s is large, and the investment schedules do not overlap. Hence investment is strictly larger in the preemptive equilibrium for a common s.

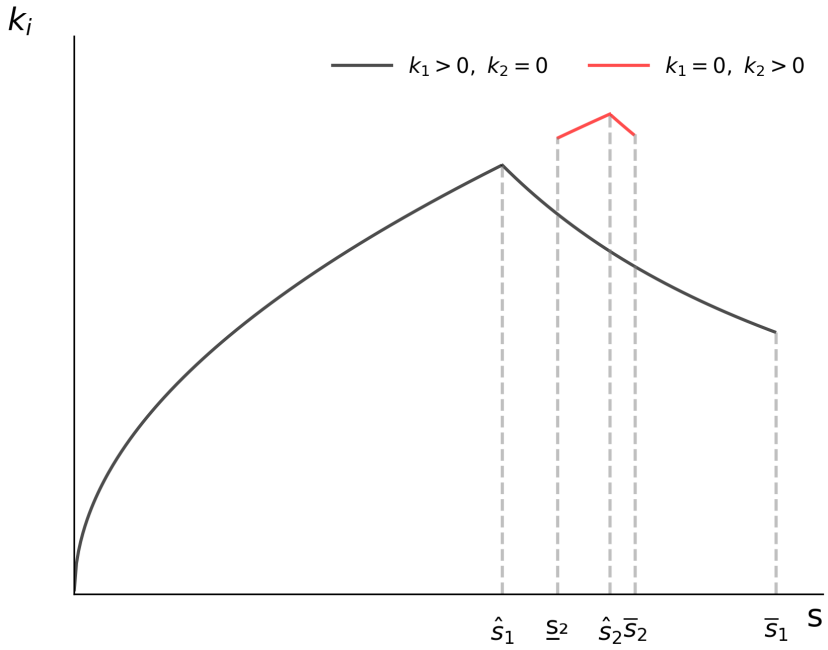

Figure 5: Investment schedules: $V_1 = 1.4, V_2 = 1, \gamma = 3$

6 Discussion

6.1 Total contest efforts

An important finding is that pre-contest investment does not necessarily preclude effort at the contest stage. Whenever the investing player sets $k_i = z(s)$, a head start is gained at the contest stage, but it is not so large that the rival simply gives up. We have also uncovered an important non-monotonicity of contest effort in s. In Figure 4, there is a contiguous interval for s that will give effort in the contest: $s \in (0, \hat{s}_1)$ if player 1 invests and $s \in [\underline{s}_2, \hat{s}_2)$ if player 2 is the investor. In Figure 5, however, the interval for s that gives fighting in the contest is disjoint. Fighting can occur for $s \in (0, \hat{s}_1)$ or $s \in [\underline{s}_2, \hat{s}_2)$ but not for the intermediate range $s \in [\hat{s}_1, \underline{s}_2)$.

From Propositions 1 and 2, let the total expected efforts in the contest when investments are $k_i = z(s), k_j = 0$ be given by $E(X \mid k_i)$ defined as the sum of the efforts of player 1 and 2 when player i is the investing player. Given parameter

combinations for which both equilibria exist, Proposition 4 shows that the expected total efforts is higher when player 2 is the investor.

Proposition 4. *If s is such that both the reinforcement and the preemptive equilibria exist, then $E(X \mid k_2) > E(X \mid k_1)$.*

It is well known that contest efforts are largest when there are small differences between the players. When player 1 invests, the initial difference between the players is exacerbated, and when 2 is the investor, it is reduced. Some of the investment by the initial underdog neutralizes the initial difference in valuations. The rest makes an advantage for player 2 at the contest stage. The amount of extra advantage that 2 finds it profitable to secure is limited by the strictly increasing, strictly convex cost of investment. Hence the rivals are more alike in a contest following investment by player 2 than by player 1. This drives the result on total expected effort at the contest stage.

6.2 Mixed-strategy equilibrium

The pure-strategy equilibria of the investment stage are limited to the range $s \in (0, \bar{s}_1]$. Beyond this, the investment stage has mixed-strategy equilibria. In the online appendix, we analyze the game with a quadratic investment cost function ($c(k) = \frac{k^2}{2}$) and characterize a set of mixed-strategy equilibria. Candidates for the pure-strategy equilibrium in this case are $k_i = \{s, \frac{V_j}{s}\}$. In this section, we highlight some key insights from that analysis, which also remain qualitatively valid with a general convex cost function.

In any of these mixed-strategy equilibria, if a player randomizes continuously over an interval of actions, then the other player must do so over an interval of the same length. Otherwise, the player with the wider support could profitably deviate by narrowing his action set without affecting his prospects in the second-period contest.

Furthermore, player 2 never chooses an action in the set $(0, \frac{V_1 - V_2}{s}]$. This is because any investment level k_2 in this range fails to improve his chances of overcoming the initial disadvantage—to push the contest toward case 4 or case 5 in Lemma 1—while incurring positive costs.

Taken together, these observations help to establish that both players employ mixed strategies that involve randomization over continuous intervals that begin strictly above zero. Furthermore, if player 1 randomizes continuously over the interval $[\delta_1, \delta_2]$, then player 2 randomizes continuously over $[\frac{V_1 - V_2}{s} + \delta_1, \frac{V_1 - V_2}{s} + \delta_2]$, for some $\delta_2 > \delta_1 > 0$.

In addition to randomizing over a continuous interval of actions, players may assign positive probability mass to discrete points of actions in equilibrium. In particular, since each player's expected payoff is non-differentiable at finitely many points, they place positive mass on a set of equidistant discrete points within the interval over which they randomize continuously.

More importantly, each player assigns strictly positive probability mass to zero. To understand why, suppose, for example, that player 1 adopts a mixed strategy that places no mass on zero and instead randomizes continuously over an interval $[\delta_1, \delta_2]$. As noted above, player 2 would then randomize continuously over $[\frac{v1-v2}{s} + \delta_1, \frac{v1-v2}{s} + \delta_2]$. However, in this case, where player 1 would place no mass at 0, player 2 would have no incentive to choose $\frac{v1-v2}{s} + \delta_1$, as this action incurs a cost without offering any positive probability of overcoming his initial disadvantage. As a result, such randomization cannot be sustained in equilibrium. A similar argument shows that player 2 must also assign strictly positive mass to zero. Ultimately, each player's incentive to choose the lowest positive investment level in the support of their mixed strategy is completely driven by the probability mass their rival assigns to zero.

This last observation has important implications for efficiency. Since both players assign positive probability mass to zero, player 2 receives zero expected payoff in any mixed-strategy equilibrium, while player 1 receives an expected payoff equal to $(V_1 - V_2)$ times the probability that player 2 assigns to zero. Therefore, all these mixed-strategy equilibria are strictly payoff-dominated by the (pure-strategy) reinforcement equilibrium whenever both types of equilibria exist.[18]

Our analysis of the game with a quadratic cost function shows that these mixed-strategy equilibria exist if s is sufficiently large and, in particular, for all $s > \bar{s}_1$ where the reinforcement equilibrium does not exist. Together with Propositions 1 and 2, this shows that an equilibrium always exists in this model.

6.3 Multiplicity of equilibria

The reinforcement and preemptive equilibria can coexist for some values of the investment efficiency parameter s. These equilibria cannot be ranked according to payoff since the identity of the player with the expected positive payoff is different in each case, while the opponent earns zero. Here, we consider the extent to which the two equilibria might be distinguished by using an equilibrium refinement.

[18] This is noted also by Hirata (2014) and used by him as an argument for precluding an analysis of mixed-strategy equilibria.

While equilibrium refinement has received much attention in the context of finite games, very few concepts have been extended to cover infinite ones. Simon and Stinchcombe (1995) have generalized the concept of trembling-hand perfection, originally introduced by Selten (1975), to infinite games. In their perfect equilibrium refinements, Simon and Stinchcombe (1995) quantify how closely each player's perturbed strategy respects the true best responses by choosing different distance measures. Weak perfection requires only that the total trembled probability mass allocated to non-best-response actions vanishes in the limit. Strong perfection enforces that the probability assigned to every individual action converges uniformly to its ideal best-response level, leaving no action with more than an infinitesimal discrepancy. Thus, the weak metric gauges accuracy with which a perturbed strategy matches the ideal best-response behavior in aggregate, while the strong metric demands exact, action-by-action accuracy.

In our model, only weak perfection appears to be tractable. In the online appendix, we examine whether weak perfection can resolve the multiplicity of pure strategy equilibria at the investment stage. We find, however, that both the reinforcement and preemptive equilibria are weakly perfect. Thus the notion of weak perfection does not help to choose between the pure-strategy equilibria when they coexist.

In the absence of a method for choosing between the equilibria at the investment stage, our results cannot be used to predict behavior. They can, on the other hand, be used as ex-post explanations of observed behavior. When a new market opportunity arises, it may well be a leading firm that makes an investment to consolidate its dominant position. On the other hand, a smaller firm may invest in a new technology that means that it is well positioned to fight for the market. Take for example the fight for dominance in the entertainment industry. As The Economist noted in 2017: "The best time to gain (or lose) audience—and to challenge the dominance of an established platform—is when technology makes a leap".[19] In 2004, Blockbuster had a dominant position in the home-entertainment industry with 9100 stores, 84 300 employees and USD 6 billion in revenue.[20] A challenger, Netflix, was founded in 1997 and invested in infrastructure that could provide customers with mail-order DVDs and a monthly flat-fee subscription for a certain number of movies. Combined with an algorithm on the Netflix home page, this effectively equipped the challenger for the contest in home entertainment. This behavior is consistent with a laggard investing to attempt to gain an advantage in a future contest, as in

[19] See The Economist, 9th February 2017

[20] See the discussion of this case in Dalton and Logan (2024).

our preemptive equilibrium. (Of course, the contest was actually fought and Netflix won).

7 Sequential investments

In this section, we change the game by assuming that the investment stage is sequential; player i invests first, this is observed by the rival who then makes an investment. A contest follows the investment stage as before. Hirata (2014) briefly discusses sequential actions in relation to his linear model, showing that if the strong player moves first, then there is a unique equilibrium in which he bids the opponents' valuation and the rival bids nothing. On the other hand, if the weak player moves first, then both bid zero. In our model, suppose that player i chooses investment level first, followed by j. When the reinforcement and preemptive equilibria both exist, they are also equilibria with sequential investment.

Proposition 5. *Suppose that* $(k_i^* > 0, k_j^* = 0)$ *is a pure-strategy equilibrium in the simultaneous-investment game. Then it is also an equilibrium in the sequential game in which player* i *invests first.*

For the ranges of s for which simultaneous-move equilibrium exist, sequential investment does not expand the possible equilibria compared to our original framework. The intuition here is that a player has a unique best response to the opponent playing 0, and the first investor can always prevent the rival from investing by playing this.

Sequential play does, however, extend the range of s for which a pure strategy equilibrium can exist. Suppose that player 1 invests first. According to Propositions 1 and 5, the reinforcement equilibrium is played for $s \in (0, \bar{s}_1]$. However, by moving first, player 1 can credibly deter investment by the rival. In the simultaneous framework, when $s > \bar{s}_1$, investment is so effective for building up a head start that player 2 has a profitable deviation from no investment. In principle, the strong player 1 can increase investment in this region, forcing the rival down to $k_2 = 0$. However, once player 2 invests nothing, player 1 will no longer want to play the costlier deterrence strategy. Moving first creates a mechanism through which player 1 can commit to deterring investment by 2. This deterrence strategy is outlined in Proposition 6. Here, we define $\check{s}$ that uniquely solves $sz(s) = V_1 + V_2$, and $\bar{\bar{s}}_i$ that uniquely solves $z(\bar{\bar{s}}_i) = c^{-1}(V_j), i \neq j$. Denote equilibrium investments when player 1 moves first by $k_i^{\times\times}$, $i = 1, 2$.

Proposition 6. *Suppose that player 1 invests first, and that this investment is observed by player 2 who then invests.*

1. *If* $\bar{s}_1 < \breve{s}$, *then* $\breve{s} < \bar{\bar{s}}_1$, *and*

   (a) $k_1^{\times\times} = \frac{M(s)-(V_1-V_2)}{s}$, $k_2^{\times\times} = 0$ *for* $s \in (\bar{s}_1, \bar{\bar{s}}_1]$;

   (b) $k_1^{\times\times} = c^{-1}(V_2) - \frac{V_1}{s}$, $k_2^{\times\times} = 0$ *for* $s \in (\bar{\bar{s}}_1, \infty)$.

2. *If* $\bar{s}_1 > \breve{s}$, *then* $k_1^{\times\times} = c^{-1}(V_2) - \frac{V_1}{s}$, $k_2^{\times\times} = 0$ *for* $s \in (\bar{s}_1, \infty)$.

In all cases, player 1 wins the contest without a fight.

Figure 6 illustrates the result in part 1 of Proposition 6, using the same example as in Figure 4.

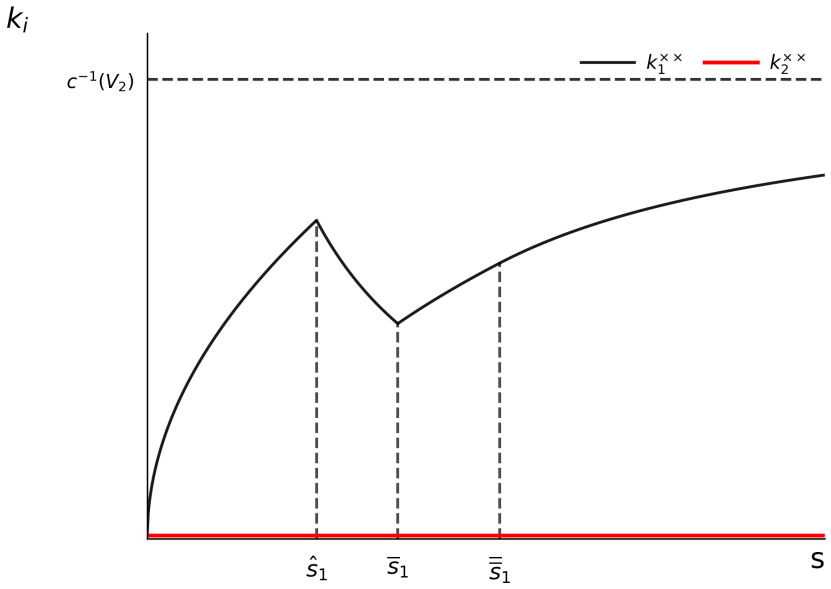

Figure 6: Player 1 invests first: $V_1 = 1.2, V_2 = 1, \gamma = 3$

For $s \in (0, \bar{s}_1]$, the equilibrium with player 1 investing first is identical to the simultaneous reinforcement equilibrium. Moving first gives player 1 a mechanism for committing to deterring the rival from making a positive investment for higher values of s. Specifically, for $s \in (\bar{s}_1, \bar{\bar{s}}_1]$, player 1 uses the deterrence investment in part 1a, and for $s \in (\bar{\bar{s}}_1, \infty)$ from part 1b. Note that this investment converges to $c^{-1}(V_2)$ as s gets large, and that the expected payoff of player 1 converges to $V_1 - V_2$.

Next, consider the sequential game in which player 2 moves first. By Propositions 3 and 5, the preemptive equilibrium arises precisely for $s \in [\underline{s}_2, \bar{s}_2]$. Outside this interval, player 2's ability to deter investment by player 1 depends on s. If

$s < \underline{s}_2$, then any $k_2 > 0$ yields player 2 a negative payoff, so he optimally chooses $k_2 = 0$, and player 1 follows with his unique best response $k_1 = \min\{z(s), \frac{V_2}{s}\}$. If $s > \bar{s}_2$, then in response to $k_2 = \frac{V_1}{s}$, player 1 can profitably deviate from zero investment. In this case, player 2 could in principle deter player 1 from investing by raising her own investment level k_2 above $\frac{V_1}{s}$. However, for sufficiently large s, any such investment-deterrent strategy becomes unprofitable, yielding a negative payoff to player 2. At this point, player 2 therefore reverts to $k_2 = 0$, inducing player 1 to play $k_1 = \frac{V_2}{s}$. The equilibrium strategy of this sequential game is summarized in Proposition 7, where we define $S := \frac{V_2}{c^{-1}(V_1) - c^{-1}(V_2)}$, and denote equilibrium investment by k_i^{**}, $i = 1, 2$.

Proposition 7. *Suppose that player 2 invests first, and that this investment is observed by player 1 who then invests.*

1. *For $s < \underline{s}_2$, $k_2^{**} = 0$, $k_1^{**} = \min\{z(s), \frac{V_2}{s}\}$.*
2. *For $s \in (\bar{s}_2, S]$, we have the following possibilities.*
   (a) *If $\bar{s}_2 < \breve{s}$, then*
      i. *For $s \in (\bar{s}_2, \min\{\bar{\bar{s}}_2, S\}]$, $k_2^{**} = \frac{M(s)+(V_1-V_2)}{s}$, $k_1^{**} = 0$.*
      ii. *For $s \in (\min\{\bar{\bar{s}}_2, S\}, S]$, which is vacuous if $\bar{\bar{s}}_2 \geq S$, $k_2^{**} = c^{-1}(V_1) - \frac{V_2}{s}$, $k_1^{**} = 0$.*
   (b) *If $\bar{s}_2 \geq \breve{s}$, then $k_2^{**} = c^{-1}(V_1) - \frac{V_2}{s}$, $k_1^{**} = 0$.*
3. *For $s > S$, $k_2^{**} = 0$, $k_1^{**} = \frac{V_2}{s}$.*

Figure 7 illustrates the equilibrium when player 2 invests first.[21] For low values of s, player 2 invests zero and player 1 follows with $k_1 = z(s)$. When $s \in [\underline{s}_2, \bar{s}_2]$, player 2 can exploit the first-mover advantage by following the strategy that is optimal also in the simultaneous case. By moving first, player 2 can actually invest for s up to S, a larger range than in the case with simultaneous investment; in the figure, this interval is divided into two portions that are continuous at $s = \bar{\bar{s}}_2$. Since the investment schedule of player 2 is increasing in s, it reaches a point (S) at which the player's expected profit is driven to zero and investment for higher s will not be profitable. Thus, player 2 reverts to $k_2 = 0$, and player 1 follows with his unique best reply $k_1 = \frac{V_2}{s}$.

When player 1 invests first, he first follows the rule that is optimal in the simultaneous game. However, he can invest for larger values of s, making the rival follow

[21] Again, we follow the example from Figure 4. For these parameter values, $S > \bar{\bar{s}}_2$.

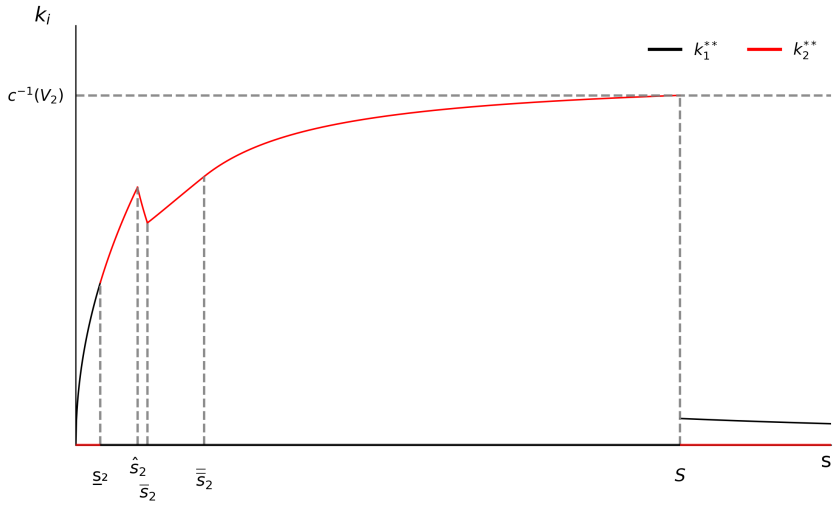

Figure 7: Player 2 invests first: $V_1 = 1.2, V_2 = 1, \gamma = 3$

with an investment of zero. When player 1 moves first, only player 1 invests. This is not the case when player 2 invests first. Again, investment rules consistent with the simultaneous game are used for $s \leq \bar{s}_2$. However, player 2 cannot profitably invest until s is sufficiently large. This means that, for low s, player 2 moves first but invests nothing, and player 1 is the investor. After this, player 2 moves first with a positive investment and the rival follows optimally by setting $k_1 = 0$. For large s, player 2 cannot exploit the first-mover advantage and player 1 is again the sole investor.

8 Concluding remarks

Rivals that fight in a contest may spend resources in preparing themselves for competition, in order to improve their chances of winning the ultimate prize. Commonly, a head start in contests is considered exogenous, being based on some history, or endogenously chosen by a contest designer to fulfill an aim. We consider asymmetric rivals who can invest in creating a head start in an all-pay auction. We have shown the existence of two types of asymmetric pure-strategy equilibria: one in which the high-valuation leader invests, and one in which the low-valuation laggard is the investor. Previous work has found that pre-contest investment will give a head start so large that there will be no further fighting at the contest stage. By extending focus to a convex cost of investment function, we successfully delineate cases in which a head start precludes fighting in the contest, and cases where it does not. One important finding that distinguishes our results from the admittedly scant literature, is that the prevalence of fighting in the contest is not necessarily

a monotonic function of the efficiency of investment. It rather depends upon the identity of the investor in equilibrium and the size of the ex-ante (dis)advantage.

In both equilibria, if there is fighting in the contest, the amount of effort expended in it is lower than the level that would prevail in a standard model without investment in the head start. The investment discourages effort in the contest, either because a player may have little incentive to try to catch up the head start, or because the head start is so large that the opponent gives up at the contest stage. In both cases, the investing player can increase the expected payoff above that from the standard all-pay auction. Hence, both players have an incentive to try to get a head start by investing prior to the contest.

This has important implications for predicting the amount of effort expected in a contest. An observer unaware of the investment opportunity may see two players with similar valuations that are competing against each other. Based on common results, the observer would predict hard fighting between almost equal rivals, resulting in the value of the prize being almost completely dissipated. However, if the players attempt to influence the game that is played by making an investment to get a head start, then this dilutes the incentives to give effort in the contest. In some cases, there will be no fighting at the contest stage at all. This is an important result for applications in which contest effort is regarded as bad, such as armed conflicts. We have shown that even a small amount of asymmetry between the players can result in quite a large difference in the maximal expected net payoffs that they can achieve. In contrast, a standard all-pay auction would give one player an expected payoff equal to the value difference (small for almost identical players), while the other expects zero.

An equilibrium exists for some parameter combinations in which the low-valuation laggard is the investing player. The investment can be used to neutralize the initial advantage of the high-value rival, and even overturn it to such an extent that the initial leader submits completely at the contest stage. This is in stark contrast to the standard all-pay auction in which the low-value player is always condemned to an expected payoff of zero. It appears that this player has a considerable incentive to try to change the rules of the standard game by investing in a head start, rather than fighting a battle that he is—at least in expectation—doomed to lose.

In common with previous literature, only one of the players will invest in a pure-strategy equilibrium. Hence the preparations made can represent some extraordinary effort such as those discussed in the Introduction. Of course, situations exist in which all players prepare for the contest, and use resources on this. Our future work on this topic will endeavor to explore this situation.

A Appendix

A.1 Proof of Lemma 3

Proof. Using Lemma 1, taking account of $k_1 \geq 0$, the expected payoff of player 1 is

$$EW_1(k_1, k_2) = \begin{cases} V_1 - c(k_1) & \text{if } k_1 \geq k_2 + \frac{V_2}{s} \\ V_1 - V_2 + s(k_1 - k_2) - c(k_1) & \text{if } k_1 \in \left(k_2 + \frac{V_2}{s} - \frac{V_1}{s},\ k_2 + \frac{V_2}{s}\right) \\ -c(k_1) & \text{if } k_1 \leq k_2 + \frac{V_2}{s} - \frac{V_1}{s}. \end{cases}$$

If $0 > k_2 + \frac{V_2}{s} - \frac{V_1}{s}$, then the third case disappears and the interval in the second one runs from 0.

To find the best response function for player 1, we look for local optima and verify that the corresponding payoff is non-negative. Note that $EW_1(k_1, k_2)$ is continuous across the boundary values for k_1.

Given k_2, it cannot be the case that $k_1 > k_2 + \frac{V_2}{s}$, since the payoff to 1 is decreasing in k in that region. Furthermore, if $k_1 \leq k_2 + \frac{V_2}{s} - \frac{V_1}{s}$, then the optimal choice is $k_1 = 0$. If $k_1 \in \left(k_2 + \frac{V_2}{s} - \frac{V_1}{s},\ k_2 + \frac{V_2}{s}\right)$, then the optimal k_1 solves $s = c'(k_1)$, i.e., $k_1 = c'^{-1}(s) = z(s)$. If $z(s)$ is in this range, then it is the optimal choice for player 1 among all $k_1 > 0$.

Denote $\hat{k}_1$ as a value of k_1 that gives a local optimum. Then

- $k_2 \leq z(s) - \frac{V_2}{s} \Rightarrow \hat{k}_1 = k_2 + \frac{V_2}{s}$;
- $k_2 \in \left(z(s) - \frac{V_2}{s},\ z(s) - \frac{V_2}{s} + \frac{V_1}{s}\right) \Rightarrow \hat{k}_1 = z(s)$;
- $k_2 \geq z(s) - \frac{V_2}{s} + \frac{V_1}{s} \Rightarrow \hat{k}_1 = 0$.

We need to check that the local optima give a non-negative payoff for player 1. We calculate

$$\begin{aligned} EW_1(z(s), k_2) &= V_1 - V_2 + sz(s) - sk_2 - c(z(s)) \\ &= s\left[z(s) - \frac{c(z(s))}{s} - \left(k_2 + \frac{V_2}{s} - \frac{V_1}{s}\right)\right] \\ &> 0 \quad \text{for} \quad k_2 < z(s) - \frac{c(z(s))}{s} - \frac{V_2}{s} + \frac{V_1}{s}. \end{aligned}$$

For the internal solution $z(s)$ to be a candidate, it must be the case that

$$z(s) - \frac{V_2}{s} \leq z(s) - \frac{c(z(s))}{s} - \frac{V_2}{s} + \frac{V_1}{s}$$

which holds for $z(s) \leq c^{-1}(V_1)$.

Recall that the best response of player i is denoted $\tilde{k}_i(k_j)$, $i, j = 1, 2, i \neq j$.

1B : $z(s) \leq c^{-1}(V_1)$

We have

$$k_2 \in \left[z(s) - \frac{V_2}{s},\ z(s) - \frac{c(z(s))}{s} - \frac{V_2 - V_1}{s} \right] \Rightarrow \tilde{k}_1(k_2) = z(s).$$

This is the second line in 1B in Lemma 3. Note that the interval is closed; the bottom of the interval represents the transfer from case 1 to case 2 in Lemma 1, which are payoff equivalent. The top of the interval gives $EW_1(z(s), k_2) = 0$ by construction.

When $k_2 \geq z(s) - \frac{c(z(s))}{s} - \frac{V_2 - V_1}{s}$, the candidate for a local optimum is $\hat{k}_1 = 0$, making this the best response; this is the third line in 1B.

Assume $k_2 \leq z(s) - \frac{V_2}{s}$, so that $\hat{k}_1 = k_2 + \frac{V_2}{s}$, giving player 1 an expected payoff of

$$\begin{aligned} EW_1\left(k_2 + \frac{V_2}{s}, k_2\right) &= V_1 - c\left(k_2 + \frac{V_2}{s}\right) \\ &\geq 0 \quad \text{for } V_1 \geq c\left(k_2 + \frac{V_2}{s}\right) \end{aligned}$$

implying that $c^{-1}(V_1) - \frac{V_2}{s} \geq k_2$.

Note that

$$c^{-1}(V_1) - \frac{V_2}{s} \geq z(s) - \frac{V_2}{s} \quad \text{for } z(s) \leq c^{-1}(V_1).$$

Since $z(s) \leq c^{-1}(V_1)$, we have

$$k_2 \leq z(s) - \frac{V_2}{s} \Rightarrow \tilde{k}_1(k_2) = k_2 + \frac{V_2}{s}.$$

This is the first line in 1B in Lemma 3.

1A : $z(s) \geq c^{-1}(V_1)$

Now, $z(s)$ is no longer a viable candidate for a maximum, and

$$k_2 \leq c^{-1}(V_1) - \frac{V_2}{s} \Rightarrow \tilde{k}_1(k_2) = k_2 + \frac{V_2}{s};$$

$$k_2 \geq c^{-1}(V_1) - \frac{V_2}{s} \Rightarrow \tilde{k}_1(k_2) = 0;$$

which are 1A in the Lemma. Note that both best responses give player 1 an expected payoff of 0 at $k_2 = c^{-1}(V_1)$.

We turn now to player 2, who, for $k_2 \geq 0$, has an expected payoff given by

$$EW_2(k_1, k_2) = \begin{cases} V_2 - c(k_2), & \text{if } k_2 \geq k_1 + \frac{V_1}{s}; \\ V_2 - V_1 + s(k_2 - k_1) - c(k_2), & \text{if } k_2 \in \left(k_1 - \frac{V_2}{s} + \frac{V_1}{s},\ k_1 + \frac{V_1}{s}\right); \\ -c(k_2), & \text{if } k_2 \leq k_1 - \frac{V_2}{s} + \frac{V_1}{s}. \end{cases}$$

Again, it is immediate that the local optima are

- $k_2 > k_1 + \frac{V_1}{s} \Rightarrow \hat{k}_2 = k_1 + \frac{V_1}{s}$
- $k_2 \leq k_1 - \frac{V_2}{s} + \frac{V_1}{s} \Rightarrow \hat{k}_2 = 0$
- $k_2 \in \left(k_1 - \frac{V_2}{s} + \frac{V_1}{s},\ k_1 + \frac{V_1}{s}\right) \Rightarrow \hat{k}_2 = z(s)$.

The expected payoff from $k_2 = z(s)$ is:

$$\begin{aligned} EW_2(k_1, z(s)) &= V_2 - V_1 + sz(s) - sk_1 - c(z(s)) \\ &= s\left[z(s) - \frac{c(z(s))}{s} - \left(k_1 + \frac{V_1}{s} - \frac{V_2}{s}\right)\right] \\ &> 0 \quad \text{for} \quad k_1 < z(s) - \frac{c(z(s))}{s} - \frac{V_1}{s} + \frac{V_2}{s}. \end{aligned}$$

Thus, we need

$$z(s) - \frac{V_1}{s} \leq z(s) - \frac{c(z(s))}{s} - \frac{V_1}{s} + \frac{V_2}{s},$$

which holds for $z(s) \leq c^{-1}(V_2)$.

2A : $z(s) \leq c^{-1}(V_2)$

Now,

$$k_1 \in \left[z(s) - \frac{V_1}{s},\ z(s) - \frac{c(z(s))}{s} - \frac{V_1 - V_2}{s}\right] \Rightarrow \tilde{k}_2(k_1) = z(s).$$

This is the second line in 2A in the Lemma.

When $k_1 \geq z(s) - \frac{c(z(s))}{s} - \frac{V_1 - V_2}{s}$, the candidate for a local optimum is $\hat{k}_2 = 0$, making this the best response; this is the third line in 2A.

Assume $k_1 \leq z(s) - \frac{V_1}{s}$, so that $\hat{k}_2 = k_1 + \frac{V_1}{s}$, giving player 2 an expected payoff of

$$\begin{aligned} EW_2\left(k_1, k_1 + \frac{V_1}{s}\right) &= V_2 - c\left(k_1 + \frac{V_1}{s}\right) \\ &\geq 0 \quad \text{for } V_2 \geq c\left(k_1 + \frac{V_1}{s}\right), \end{aligned}$$

implying that $c^{-1}(V_2) - \frac{V_1}{s} \geq k_1$. Note that

$$c^{-1}(V_2) - \frac{V_1}{s} \geq z(s) - \frac{V_1}{s} \quad \text{for } z(s) \leq c^{-1}(V_2).$$

Since $z(s) \leq c^{-1}(V_2)$, we have

$$k_1 \leq z(s) - \frac{V_1}{s} \Rightarrow \tilde{k}_2(k_1) = k_1 + \frac{V_1}{s}.$$

This is the first line in 2A in Lemma 3.

2B : $z(s) \geq c^{-1}(V_2)$

Now, $z(s)$ is no longer a viable candidate for a maximum, and we have

$$k_1 \leq c^{-1}(V_2) - \frac{V_1}{s} \Rightarrow \tilde{k}_2(k_1) = k_1 + \frac{V_1}{s};$$

$$k_1 \geq c^{-1}(V_1) - \frac{V_1}{s} \Rightarrow \tilde{k}_2(k_1) = 0;$$

which are 2B in the Lemma.

□

A.2 Proof of Proposition 1

Proof. It follows from Lemma 3 that, if player 1 is the only investor, he will choose $k_1 \in \{z(s), \frac{V_2}{s}\}$.

Part 1: Consider the strategy profile $(k_1 = z(s), k_2 = 0)$. By Lemma 3 (case 1B), we have $\tilde{k}_1(0) = z(s)$ if and only if

$$\text{(1.a)} \quad z(s) \leq c^{-1}(V_1), \quad \text{(1.b)} \quad z(s) \leq \frac{V_2}{s}, \quad \text{(1.c)} \quad M(s) + V_1 - V_2 \geq 0.$$

Assuming (1.a)–(1.c) are met, it follows from Lemma 3 (cases 2A and 2B) that $\tilde{k}_2(z(s)) = 0$ if and only if one of the following two sets of conditions are met:

$$\text{(1.d.i)} \quad z(s) \leq c^{-1}(V_2), \quad \text{and} \quad \text{(1.d.ii)} \quad c(z(s)) \geq V_2 - V_1; \text{ or}$$
$$\text{(1.e.i)} \quad c^{-1}(V_2) < z(s) \leq c^{-1}(V_1), \quad \text{and} \quad \text{(1.e.ii)} \quad z(s) \geq c^{-1}(V_2) - \frac{V_1}{s}.$$

Since, (1.d.ii) always holds, and (1.e.i) $\Longrightarrow$ (1.e.ii), we conclude that $\tilde{k}_2(z(s)) = 0$ whenever (1.a) is satisfied.

Next, we claim that (1.b) and (1.c) imply (1.a). To see this, rewrite (1.c) as $[sz(s) - V_2] + [V_1 - c(z(s))] \geq 0$. Since the first term is negative by (1.b), the second

term must be positive, which implies (1.a). Furthermore, (1.c) always holds, since $M(s) \geq 0$.

Therefore, we conclude that $(k_1 = z(s), k_2 = 0)$ is an equilibrium if (1.b) holds, or, equivalently, if $0 < s \leq \hat{s}_1$. In this case, $\Delta_1 = sz(s) \leq V_2$. It follows from Lemma 1 that, if $\Delta_1 < V_2$, which is when $0 < s < \hat{s}_1$, then the continuation game moves to case 2 of Lemma 1, in which case players exert positive efforts in the second-period contest and net payoffs of player 1 and player 2 are given by $V_1 - V_2 + M(s)$ and 0, respectively. If $\Delta_1 = sz(s) = V_2$, implying $k_1 = z(s) = \frac{V_2}{s}$, then the continuation game moves to case 1 of Lemma 1; we consider this case in the analysis of part 2 below.

Part 2: Consider the strategy profile $(k_1 = \frac{V_2}{s}, k_2 = 0)$. There are two cases to consider, depending upon which parts of Lemma 3 support this profile as best responses.

Case 2.I. It follows from cases 1A and 2B of Lemma 3 that we can have $\tilde{k}_1(0) = \frac{V_2}{s}$ and $\tilde{k}_2(\frac{V_2}{s}) = 0$ if

$$\text{(2.I.a)} \quad z(s) \geq c^{-1}(V_1) \geq \frac{V_2}{s}, \quad \text{(2.I.b)} \quad \frac{V_2}{s} \geq c^{-1}(V_2) - \frac{V_1}{s}.$$

Note that we can rewrite (2.I.b) as $s \leq \frac{V_1+V_2}{c^{-1}(V_2)}$.

Case 2.II. It follows from cases 1B, 2A, and 2B of Lemma 3 that we can have $\tilde{k}_1(0) = \frac{V_2}{s}$ and $\tilde{k}_2(\frac{V_2}{s}) = 0$ if one of the following two sets of conditions are met:

$$\text{(2.II.a)} \quad \frac{V_2}{s} \leq z(s) \leq c^{-1}(V_2), \quad \text{and} \quad \text{(2.II.b)} \quad 0 \geq z(s) - \frac{c^{-1}(V_1)}{s} - \frac{V_1}{s}; \text{ or}$$
$$\text{(2.II.c)} \quad \max\{c^{-1}(V_2), \frac{V_2}{s}\} < z(s) \leq c^{-1}(V_1), \quad \text{and} \quad \text{(2.II.d)} \quad \frac{V_2}{s} \geq c^{-1}(V_2) - \frac{V_1}{s}.$$

Note that (2.II.b) can be rewritten as $s \leq M^{-1}(V_1)$, and (2.II.d) can be rewritten as $s \leq \frac{V_1+V_2}{c^{-1}(V_2)}$.

Since $sk - c(k) > 0$ for all $k \in (0, z(s)]$, we have that $z(s) \geq c^{-1}(V_2) \implies$

$$c^{-1}(V_2) > \frac{V_2}{s} \tag{A.1}$$

We can therefore express (2.II.c) as $\frac{V_2}{s} < c^{-1}(V_2) < z(s) \leq c^{-1}(V_1)$.

We combine the findings from case 2.I. and case 2.II. as follows: If $\frac{V_2}{s} \leq z(s) < c^{-1}(V_2)$, then $(k_1 = \frac{V_2}{s}, k_2 = 0)$ is an equilibrium if $s \leq M^{-1}(V_1)$; and if $\frac{V_2}{s} \leq c^{-1}(V_2) \leq z(s)$, then $(k_1 = \frac{V_2}{s}, k_2 = 0)$ is an equilibrium if $s \leq \frac{V_1+V_2}{c^{-1}(V_2)}$.

Furthermore, since $z(s)$ increases with s, we can choose the appropriate upper bound by examining if $z(\frac{V_1+V_2}{c^{-1}(V_2)}) \leq c^{-1}(V_2)$, which allows us to conclude that $(k_1 =$

$\frac{V_2}{s}, k_2 = 0)$ is an equilibrium for $s \leq \frac{V_1+V_2}{c^{-1}(V_2)}$ if $z(\frac{V_1+V_2}{c^{-1}(V_2)}) > c^{-1}(V_2)$, and for $s \leq M^{-1}(V_1)$ if $z(\frac{V_1+V_2}{c^{-1}(V_2)}) \leq c^{-1}(V_2)$. In this equilibrium, $\Delta_1 = V_2$, and the second-period contest moves to case 1 of Lemma 1. It directly follows from Lemma 1 that player 1's net payoff is given by $V_1 - c(\frac{V_2}{s})$, and that there is no effort in the period-2 contest. Note that the expected payoff $V_1 - c(\frac{V_2}{s}) > V_1 - V_2$ by (A.1).

To complete the proof, we show that $\bar{s}_1 > \hat{s}_1$. Suppose first that $\bar{s}_1$ is given by the first line in (1), implying that $c(z(\bar{s}_1)) > V_2 = \hat{s}_1 z(\hat{s}_1)$, where the equality follows from the definition of $\hat{s}_1$. Subtracting $c(z(\hat{s}_1))$ from both sides gives

$$c(z(\bar{s}_1)) - c(z(\hat{s}_1)) > \hat{s}_1 z(\hat{s}_1) - c(z(\hat{s}_1)) = M(\hat{s}_1) \geq 0.$$

Hence, $c(z(\bar{s}_1)) - c(z(\hat{s}_1)) > 0$ which implies $\bar{s}_1 > \hat{s}_1$, since $c()$ and $z()$ are both strictly increasing functions.

Suppose next that $\bar{s}_1$ is given by the second line in (1). Then $V_1 = \bar{s}_1 z(\bar{s}_1) - c(z(\bar{s}_1))$. Using the definition of $\hat{s}_1$, we can write

$$\bar{s}_1 z(\bar{s}_1) - c(z(\bar{s}_1)) - \hat{s}_1 z(\hat{s}_1) = V_1 - V_2 > 0.$$

This necessarily requires $\bar{s}_1 z(\bar{s}_1) - \hat{s}_1 z(\hat{s}_1) > 0$, which implies $\bar{s}_1 > \hat{s_1}$, since $sz(s)$ is strictly increasing in s.

□

A.3 Proof of Proposition 2

Proof. If player 2 is the only investor, then, by Lemma 3, his optimal choice $k_2 \in \{z(s), \frac{V_1}{s}\}$.

Part 1: Consider the strategy profile $(k_1 = 0, k_2 = z(s))$. By Lemma 3 (case 2A), we have $\tilde{k}_2(0) = z(s)$ if and only if

$$\text{(1.a)} \quad z(s) \leq c^{-1}(V_2), \quad \text{(1.b)} \quad z(s) \leq \frac{V_1}{s}, \quad \text{(1.c)} \quad M(s) - V_1 + V_2 \geq 0.$$

Assuming (1.a)—(1.c) are met, it follows from Lemma 3 (case 1B) that $\tilde{k}_1(z(s)) = 0$ if and only if

$$\text{(1.d)} \quad c(z(s)) \geq V_1 - V_2.$$

Observe that (1.b) and (1.c) imply (1.a). This is because we can express (1.c) as $[sz(s) - V_1] + [V_2 - c(z(s))] \geq 0$, and since the first term is negative by (1.b), the second term must be positive, which implies (1.a).

Since (1.b) $\implies$ $s \leq \hat{s}_1$, we have that $(k_1 = 0, k_2 = z(s))$ is an equilibrium if both (1.c) and (1.d) hold, or, equivalently, if $\underline{s}_2 \leq s \leq \hat{s}_2$.

In this equilibrium, $\Delta_1 = -sz(s)$. Since, $M(s) \geq V_1 - V_2$, it follows that $sz(s) \geq V_1 - V_2$, or, equivalently, $\Delta_1 \leq -(V_1 - V_2)$.

Furthermore , it follows from Lemma 1 that, if $\Delta_1 > -V_1$, which is when $0 < s < \hat{s}_2$, then the continuation game moves to case 4 of Lemma 1, in which case players exert positive efforts in the second-period contest and the net payoffs of player 1 and player 2 are given by 0 and $V_2 - V_1 + M(s)$, respectively. If $\Delta_1 = -sz(s) = -V_1$, implying $k_2 = z(s) = \frac{V_1}{s}$, then the continuation game moves to case 5 of Lemma 1, and we will consider this case in the analysis of part 2 below.

Part 2: Consider the strategy profile $(k_1 = 0, k_2 = \frac{V_1}{s})$. There are two cases to consider.

Case 2.I. It follows from cases 2A and 1B of Lemma 3 that we can have $\tilde{k}_1(\frac{V_1}{s}) = 0$ and $\tilde{k}_2(0) = \frac{V_1}{s}$ if

$$\text{(2.I.a)} \quad c^{-1}(V_2) \geq z(s) \geq \frac{V_1}{s}, \quad \text{(2.I.b)} \quad M(s) \leq V_2.$$

Case 2.II. It follows from cases 1A, 1B, and 2B of Lemma 3 that we can have $\tilde{k}_1(\frac{V_1}{s}) = 0$ and $\tilde{k}_2(0) = \frac{V_1}{s}$ if one of the following two sets of conditions are met:

$$\text{(2.II.a)} \quad c^{-1}(V_1) \geq z(s) > c^{-1}(V_2) \geq \frac{V_1}{s}, \quad \text{and} \quad \text{(2.II.b)} \quad M(s) \leq V_2; \text{ or}$$

$$\text{(2.II.c)} \quad z(s) > c^{-1}(V_1) \geq c^{-1}(V_2) \geq \frac{V_1}{s}, \quad \text{and} \quad \text{(2.II.d)} \quad \frac{V_1}{s} \geq c^{-1}(V_1) - \frac{V_2}{s}.$$

Note that (2.II.b) can be rewritten as $s \leq M^{-1}(V_2)$, and (2.II.d) can be rewritten as $s \leq \frac{V_1+V_2}{c^{-1}(V_1)}$. We therefore combine the findings from case 2.I. and case 2.II. as follows: If $\frac{V_1}{s} \leq z(s) \leq c^{-1}(V_1)$, then we have that $(k_1 = 0, k_2 = \frac{V_1}{s})$ is an equilibrium if $s \leq M^{-1}(V_2)$; and if $\frac{V_2}{s} \leq c^{-1}(V_1) < z(s)$, then we have that $(k_1 = 0, k_2 = \frac{V_1}{s})$ is an equilibrium if $s \leq \frac{V_1+V_2}{c^{-1}(V_1)}$. Furthermore, since $z(s)$ increases in s, we can choose the appropriate upper bound by examining whether $z(\frac{V_1+V_2}{c^{-1}(V_1)}) \leq c^{-1}(V_1)$, and we conclude that $(k_1 = 0, k_2 = \frac{V_1}{s})$ is an equilibrium for $s \leq \frac{V_1+V_2}{c^{-1}(V_1)}$ if $z(\frac{V_1+V_2}{c^{-1}(V_1)}) > c^{-1}(V_1)$, and for $s \leq M^{-1}(V_2)$ if $z(\frac{V_1+V_2}{c^{-1}(V_1)}) \leq c^{-1}(V_1)$. In this equilibrium, $\Delta_1 = -V_1$, and the second-period contest moves to case 5 of Lemma 1. It directly follows from Lemma 1 that player 2's net payoff is given by $V_2 - c(\frac{V_1}{s})$, and that there are no efforts in the period-2 contest. □

A.4 Proof of Proposition 3

Proof. **Part 1:** To prove the first part, we show that $\min\{\underline{s}_2, \hat{s}_2\} \leq \bar{s}_2$ if and only if $M(\hat{s}_2) \leq V_2$.

(If part): Assume that $M(\hat{s}_2) \leq V_2$, or, equivalently, $\hat{s}_2 \leq M^{-1}(V_2)$; we are to show that $\min\{\underline{s}_2, \hat{s}_2\} \leq \bar{s}_2$.

If $\bar{s}_2 = M^{-1}(V_2)$, then the result follows immediately, since $\min\{\underline{s}_2, \hat{s}_2\} \leq \hat{s}_2 \leq \bar{s}_2$. Now, consider the case where $z\big(\frac{V_1+V_2}{c^{-1}(V_1)}\big) > c^{-1}(V_1)$, so that $\bar{s}_2 = \frac{V_1+V_2}{c^{-1}(V_1)}$. Since, for any s, the inequality $sk - c(k) \leq M(s)$ holds for all $k \geq 0$, we can take $s = M^{-1}(V_2)$ and $k = c^{-1}(V_1)$ to obtain: $M^{-1}(V_2)c^{-1}(V_1) - V_1 \leq V_2$, or, equivalently, $M^{-1}(V_2) \leq \bar{s}_2$. Hence, $\min\{\underline{s}_2, \hat{s}_2\} \leq \hat{s}_2 \leq M^{-1}(V_2) \leq \bar{s}_2$.

(Only if part): Assume that $\min\{\underline{s}_2, \hat{s}_2\} \leq \bar{s}_2$; we are to show that $M(\hat{s}_2) \leq V_2$. There are two sub-cases to consider.

Case 1: $\underline{s}_2 \leq \hat{s}_2$ and $\underline{s}_2 \leq \bar{s}_2$. Since $\underline{s}_2 \leq \hat{s}_2$, we must have $c(z(\hat{s}_2)) \geq V_1 - V_2$. Therefore, $M(\hat{s}_2) = \hat{s}_2 z(\hat{s}_2) - c(z(\hat{s}_2)) = V_1 - c(z(\hat{s}_2)) \leq V_2$.

Case 2: $\hat{s}_2 < \underline{s}_2$ and $\hat{s}_2 \leq \bar{s}_2$. If $\bar{s}_2 = M^{-1}(V_2)$, then it follows immediately that $M(\hat{s}_2) \leq V_2$.

Now, consider the case where $z\big(\frac{V_1+V_2}{c^{-1}(V_1)}\big) > c^{-1}(V_1)$, so that $\bar{s}_2 = \frac{V_1+V_2}{c^{-1}(V_1)}$. Define $s^0 := z^{-1}(c^{-1}(V_1))$.

First, we claim that $\hat{s}_2 \leq s^0$. To see this, suppose to the contrary that $s^0 < \hat{s}_2$, or, equivalently, $c^{-1}(V_1) < z(\hat{s}_2)$. Since $sk - c(k) > 0$ for all $0 < k < z(s)$, we then have that $\hat{s}_2 c^{-1}(V_1) - V_1 > 0 \implies c^{-1}(V_1) > \frac{V_1}{\hat{s}_2} = z(\hat{s}_2)$, which leads to a contradiction. Therefore, $\hat{s}_2 \leq s^0$, and, consequently, $M(\hat{s}_2) \leq M(s^0)$.

Next, we claim that $M(s^0) \leq V_2$, or, equivalently, $s^0 \leq M^{-1}(V_2)$. Since we now deal with the case $\frac{V_1+V_2}{c^{-1}(V_1)} > z^{-1}(c^{-1}(V_1)) = s^0$, it follows that $V_2 > s^0 c^{-1}(V_1) - V_1 = M(s^0)$. Therefore, $M(\hat{s}_2) \leq M(s^0) < V_2$.

Part 2: To prove the second part, we show that $\bar{s}_2 \leq \bar{s}_1$.

Claim 1: $M^{-1}(V_1) \leq \dfrac{V_1 + V_2}{c^{-1}(V_2)}$, and $M^{-1}(V_2) \leq \dfrac{V_1 + V_2}{c^{-1}(V_1)}$.

Proof of Claim 1: By the definition of $M(s)$, we know that, for any s and for all $k \geq 0$, we have $sc^{-1}(V_2) - V_2 \leq M(s)$. Taking $s = M^{-1}(V_1)$ and $k = c^{-1}(V_2)$, we get $M^{-1}(V_1)c^{-1}(V_2) - V_2 \leq V_1$, which gives the first inequality. To prove the second, consider $s = M^{-1}(V_2)$ and $k = c^{-1}(V_1)$. This proves Claim 1.

Since $M^{-1}(\cdot)$ and $c^{-1}(\cdot)$ are increasing functions, we will therefore have one of the two following scenarios:

Case 1: $M^{-1}(V_2) \leq \dfrac{V_1 + V_2}{c^{-1}(V_1)} \leq M^{-1}(V_1) \leq \dfrac{V_1 + V_2}{c^{-1}(V_2)}$;

Case 2: $M^{-1}(V_2) \leq M^{-1}(V_1) \leq \dfrac{V_1 + V_2}{c^{-1}(V_1)} \leq \dfrac{V_1 + V_2}{c^{-1}(V_2)}$.

In Case 1, it follows immediately that $\bar{s}_2 \leq \bar{s}_1$. So, consider Case 2, and assume that $\bar{s}_2 = \frac{V_1+V_2}{c^{-1}(V_1)}$, which happens only if $z(\frac{V_1+V_2}{c^{-1}(V_1)}) > c^{-1}(V_1)$. Then, $z(\frac{V_1+V_2}{c^{-1}(V_2)}) > z(\frac{V_1+V_2}{c^{-1}(V_1)}) > c^{-1}(V_1) > c^{-1}(V_2)$, and so $\bar{s}_1 = \frac{V_1+V_2}{c^{-1}(V_2)}$, which is higher than $\bar{s}_2$. This completes the proof of Proposition 3. □

A.5 Proof of Proposition 4

Proof. Write $sz(s) := h$. We have

$$E(X \mid k_1) \;=\; \frac{1}{2V_2}\big(V_2 - h\big)^2 \;+\; \frac{1}{2V_1}\big(V_2^2 - h^2\big),$$

$$E(X \mid k_2) \;=\; \frac{V_1}{2V_2}\big(V_1 - h\big) \;+\; \frac{1}{2V_1}\big(V_1 - h\big)^2.$$

We can calculate the difference as

$$\begin{aligned} &E(X \mid k_2) - E(X \mid k_1) \\ =&\underbrace{\frac{1}{2}(V_1 - V_2)\frac{(V_1+V_2)^2}{V_1V_2}}_{A} + \underbrace{h\left(\frac{2V_2 - V_1}{2V_2}\right)}_{B} + \underbrace{h^2\left(\frac{2V_2 - V_1}{2V_1V_2}\right)}_{C}. \end{aligned}$$

$A > 0,\ B > 0,\ C > 0$ follows directly from $V_1 > V_2 > \frac{V_1}{2}$. □

A.6 Proof of Proposition 5

Proof. Suppose that $(k_i^*, 0)$ is an equilibrium in the simultaneous investment game, i.e. they are mutual best replies. Suppose further that player i moves first in the sequential game and deviates to $\hat{k}_i \neq k_i^*$, and that player j has a best reply $\hat{k}_j = BR(\hat{k}_j)$. The deviation by player i is profitable if

$$(i) \quad EW_i(\hat{k}_i, \hat{k}_j) > EW_i(k_i^*, 0).$$

However, it must also be the case that

$$(ii) \quad EW_i(k_i^*, 0) \geq EW_i(\hat{k}_i, 0) \geq EW_i(\hat{k}_i, \hat{k}_j) > EW_i(k_i^*, 0)$$

, which is a contradiction. The first inequality in (ii) is from the definition of k_i^* as a best response to $k_j^* = 0$, and the second inequality comes from the fact that i's payoff is decreasing in k_j. □

A.7 Proof of Proposition 6

Proof. There are two cases in which the reinforcement equilibrium can break down with simultaneous play.

$$(i) \quad s > M^{-1}(V_1) \quad \text{and} \quad z(s) < \frac{V_1 + V_2}{s};$$
$$(ii) \quad s > \frac{V_1 + V_2}{c^{-1}(V_2)} \quad \text{and} \quad z(s) > \frac{V_1 + V_2}{s}.$$

In (i), $\bar{s}_1 = M^{-1}(V_1)$, and for $s > \bar{s}_1$, the simultaneous-move equilibrium breaks down since player 2 has a feasible and profitable deviation to $k_2 = \min\{z(s), k_1 + \frac{V_1}{s}\}$. The investment level in part 1 of the proposition deters this deviation. In (ii), $\bar{s}_1 = \frac{V_1+V_2}{c^{-1}(V_2)}$, and for $s > \bar{s}_1$, the feasible, profitable deviation is to $k_2 = k_1 + \frac{V_1}{s}$, which is deterred by player 1 in part 2 of the proposition.

In the following claim, we argue that there can only be two possibilities depending on the cost function.

Claim 1: We can have one of the following two cases:

- Case 1: $M^{-1}(V_1) < \frac{V_1+V_2}{c^{-1}(V_2)} < \breve{s}$.
- Case 2: $\breve{s} \leq M^{-1}(V_1) \leq \frac{V_1+V_2}{c^{-1}(V_2)}$.

Proof of Claim 1: First, note that $M^{-1}(V_1) \leq \frac{V_1+V_2}{c^{-1}(V_2)}$, which follows from part 2 of the proof of Proposition 3.

Now, we show that $M^{-1}(V_1) < \breve{s} < \frac{V_1+V_2}{c^{-1}(V_2)}$ cannot occur.

This is because $\frac{V_1+V_2}{c^{-1}(V_2)} > \breve{s} \implies z(\breve{s}) > c^{-1}(V_2) \implies \breve{s}z(\breve{s}) - V_2 > \breve{s}z(\breve{s}) - c\big(z(\breve{s})\big)$. Applying $\breve{s}z(\breve{s}) = V_1 + V_2$, this further reduces to $V_1 > M(\breve{s}) \implies \breve{s} < M^{-1}(V_1)$.

Therefore, we can either have $M^{-1}(V_1) < \frac{V_1+V_2}{c^{-1}(V_2)} < \breve{s}$, or $\breve{s} \leq M^{-1}(V_1) \leq \frac{V_1+V_2}{c^{-1}(V_2)}$. This completes the proof of Claim 1.

We first consider Part 1. In this case, $\bar{s}_1 = M^{-1}(V_1)$, and player 2 has a feasible and profitable deviation $k_2 = \min\{z(s), k_1 + \frac{V_1}{s}\}$ for s above $\bar{s}_1$.

To deter player 2 from investing $z(s)$, player 1 can set the lowest k_1 so that

$$EW_2(k_1, z(s)) = V_2 - V_1 + s(z(s) - k_1) - c(z(s)) \leq 0,$$

which gives a deterrence investment of

$$k_1^d = \frac{M(s) - (V_1 - V_2)}{s}.$$

This is a feasible deterrent strategy for player 1 if (i) $z(s) \leq k_1^d + \frac{V_1}{s}$, and (ii) $EW_1(k_1^d, 0) = V_1 - c(k_1^d) > 0$. The inequality in (i) ensures that player 2 prefers $z(s)$ over $k_1^d + \frac{V_1}{s}$ while the inequality in (ii) ensures that player 1 receives a positive profit by playing the deterrence investment strategy.

Note that $z(s) \leq k_1^d + \frac{V_1}{s} \implies z(s) \leq \frac{M(s)+V_2}{s} \implies z(s) \leq c^{-1}(V_2) \implies s \leq \bar{\bar{s}}_1$. Furthermore, since k_1^d is strictly increasing in s, $EW_1(k_1^d, 0) = V_1 - c(k_1^d)$ decreases with s, and therefore the inequality in (ii) holds for all $s \leq \bar{\bar{s}}_1$ if it holds at $\bar{\bar{s}}_1$; we consider the case of $s > \bar{\bar{s}}_1$ below.

At $s = \bar{\bar{s}}_1$, $k_1^d = \frac{M(s)-V_1+V_2}{s} = z(\bar{\bar{s}}_1) - \frac{V_1}{\bar{\bar{s}}_1}$, since $c(z(\bar{\bar{s}}_1)) = V_2$. Furthermore, since

$$c^{-1}(V_1) > c^{-1}(V_2) = z(\bar{\bar{s}}_1) > z(\bar{\bar{s}}_1) - \frac{V_1}{\bar{\bar{s}}_1} = k_1^d,$$

it follows that $EW_1(k_1^d, 0) = V_1 - c(k_1^d) > 0$. Therefore, for all $s \in (\bar{s}_1, \bar{\bar{s}}_1]$, $k_1 = \frac{M(s)-(V_1-V_2)}{s}$, $k_2 = 0$ constitute an equilibrium strategy profile.

Next, consider $s > \bar{\bar{s}}_1$. In this case, for any investment level $k_1 \geq k_1^d$ by player 1, player 2 responds by investing $k_2 = k_1 + \frac{V_1}{s}$. To deter player 2 from investing, player 1 sets the lowest k_1 so that

$$EW_2\left(k_1, k_1 + \frac{V_1}{s}\right) = V_2 - c\left(k_1 + \frac{V_1}{s}\right) \leq 0,$$

which gives a deterrence investment of

$$k_1^D = c^{-1}(V_2) - \frac{V_1}{s}.$$

This investment level (and hence its cost) is increasing in s. This means that if $\lim_{s\to\infty} EW_1(k_1^D, 0) > 0$, then it is positive for smaller s. It follows that

$$\lim_{s\to\infty} EW_1(k_1^D, 0) = \lim_{s\to\infty} \left[V_1 - c(k_1^D)\right] = V_1 - V_2 > 0.$$

Therefore, $k_1 = c^{-1}(V_2) - \frac{V_1}{s}$, $k_2 = 0$ constitute an equilibrium strategy profile for $s > \bar{\bar{s}}_1$.

We next consider Part 2. In this case, $\bar{s}_1 = \frac{V_1+V_2}{c^{-1}(V_2)}$, and player 2 has a feasible and profitable deviation $k_2 = k_1 + \frac{V_1}{s}$ for s above $\bar{s}_1$. This would give player 2 a sure win in the contest. However, as we have just shown above, player 1 can deter this by setting $k_1 = k_1^D$, and continues to receive a positive payoff for all s above $\bar{s}_1$. □

A.8 Proof of Proposition 7

Proof. **Part 1.** Consider $s < \underline{s}_2$. Since both $M(s)$ and $c(z(s))$ are increasing in s, it follows from the definition of $\underline{s}_2$ in (2) that, for all $s < \underline{s}_2$,

$$M(s) < V_1 - V_2 \iff V_2 - V_1 + M(s) < 0. \tag{A.2}$$

We now show that player 2 receives a negative payoff from choosing any investment $k_2 > 0$. The expected payoff for 2 is given by

$$EW_2(BR_1(k_2), k_2) = \begin{cases} V_2 - V_1 + s\left(k_2 - BR_1(k_2)\right) - c(k_2) & \text{if } k_2 < BR_1(k_2) + \frac{V_1}{s} \\ V_2 - c(k_2) & \text{if } k_2 \geq BR_1(k_2) + \frac{V_1}{s} \end{cases}. \tag{A.3}$$

Consider the first line in (A.3). Since $M(s) \geq sk_2 - c(k_2)$ for all $k_2 \geq 0$, we have

$$V_2 - V_1 + s\left(k_2 - BR_1(k_2)\right) - c(k_2) \leq V_2 - V_1 + M(s) - sBR_1(k_2) < 0,$$

where the final inequality follows from (A.2).

Next, consider the second line in (A.3). Applying the fact that $M(s) > \frac{sV_1}{s} - c\left(\frac{V_1}{s}\right) = V_1 - c\left(\frac{V_1}{s}\right)$ and $c(\cdot)$ is an increasing function, we get, for all $k_2 \geq BR_1(k_2) + \frac{V_1}{s}$,

$$V_2 - c(k_2) \leq V_2 - c(BR_1(k_2) + \frac{V_1}{s}) < V_2 - c\left(\frac{V_1}{s}\right) < V_2 - V_1 + M(s) < 0,$$

where the final inequality follows from (A.2).

Hence, any $k_2 > 0$ gives a strictly negative expected profit and is dominated by $k_2 = 0$. From Lemma 3, player 1's unique best response to this is $k_1 = \min\{z(s), \frac{V_2}{s}\}$ as claimed.

Part 2. We first consider how $\bar{s}_2$ is determined in (3). Claim 1 in the proof of Proposition 3 shows that

$$M^{-1}(V_2) \leq \frac{V_1 + V_2}{c^{-1}(V_1)}.$$

Now we show that $M^{-1}(V_2) < \breve{s} < \frac{V_1+V_2}{c^{-1}(V_1)}$ cannot occur. By applying $\breve{s}z(\breve{s}) = V_1 + V_2$, we get the following inequalities: $\frac{V_1+V_2}{c^{-1}(V_1)} > \breve{s} \implies z(\breve{s}) > c^{-1}(V_1) \implies -V_1 > -c(z(\breve{s})) \implies (V_1+V_2) - V_1 > \breve{s}z(\breve{s}) - c\big(z(\breve{s})\big) \implies V_2 > M(\breve{s}) \implies M^{-1}(V_1) > \breve{s}$. Therefore the ordering $M^{-1}(V_2) < \breve{s} < \frac{V_1+V_2}{c^{-1}(V_1)}$ cannot occur.

Only two possible orderings remain:

- Case 1: $M^{-1}(V_2) < \frac{V_1+V_2}{c^{-1}(V_1)} < \breve{s}$. The deviation analysis that underpins (3) then sets $\bar{s}_2 = M^{-1}(V_2)$.

- Case 2: $\breve{s} \leq M^{-1}(V_2) \leq \frac{V_1+V_2}{c^{-1}(V_1)}$. It follows from the deviation analysis characterizing (3) that $\bar{s}_2 = \frac{V_1+V_2}{c^{-1}(V_1)}$.

We begin with Case 1, where $\bar{s}_2 = M^{-1}(V_2)$, and player 1 has a feasible and profitable deviation $k_1 = \min\{z(s), k_2 + \frac{V_2}{s}\}$ for s above $\bar{s}_2$.

To deter player 1 from investing $z(s)$, player 2 can set the lowest k_2 so that

$$EW_1(z(s), k_2) = V_1 - V_2 + M(s) - k_2 \leq 0,$$

which gives a deterrence investment of

$$k_2^d = \frac{M(s) + V_1 - V_2}{s}.$$

This is a feasible deterrent strategy for player 2 if (i) $z(s) \leq k_2^d + \frac{V_2}{s}$, and (ii) $EW_2(0, k_2^d) = V_2 - c(k_2^d) > 0$. The inequality in (i) ensures that player 1 prefers $z(s)$ over $k_2^d + \frac{V_2}{s}$ while the inequality in (ii) ensures that player 2 receives a positive profit by playing the deterrence investment strategy.

Note that $z(s) \leq k_2^d + \frac{V_2}{s} \implies z(s) \leq \frac{M(s)+V_1}{s} \implies z(s) \leq c^{-1}(V_1) \implies s \leq \bar{\bar{s}}_2$. Furthermore, since k_2^d is strictly increasing in s, $EW_2(0, k_2^d) = V_2 - c(k_2^d)$ decreases with s, and therefore the inequality in (ii) holds for all $s \leq \bar{\bar{s}}_2$ if it holds at $\bar{\bar{s}}_2$; we consider the case of $s > \bar{\bar{s}}_2$ below.

At $s = \bar{\bar{s}}_2$, $k_2^d = \frac{M(\bar{\bar{s}}_2)+V_1-V_2}{\bar{\bar{s}}_2} = z(\bar{\bar{s}}_2) - \frac{V_2}{\bar{\bar{s}}_2} = c^{-1}(V_1) - \frac{V_2}{\bar{\bar{s}}_2}$, since $c(z(\bar{\bar{s}}_2)) = V_1$. Furthermore,

$$EW_2(0, k_2^d) = V_2 - c(k_2^d),$$

which is positive if $k_2^d \leq c^{-1}(V_2) \iff c^{-1}(V_1) - \frac{V_2}{\bar{\bar{s}}_2} \leq c^{-1}(V_2) \iff \bar{\bar{s}}_2 \leq S$.

If $\bar{\bar{s}}_2 > S$, then player 2 can play the investment deterrent strategy k_2^d for $s \leq S$, and for $s > S$, we derive the equilibrium strategy profile later in Part 3.

If $\bar{\bar{s}}_2 \leq S$, then $k_1 = 0, k_2 = k_2^d$ constitutes an equilibrium at the investment stage for $s \in (\bar{s}_2, \bar{\bar{s}}_2]$.

For $s > \bar{\bar{s}}_2$, we have that, for any investment level $k_2 \geq k_2^d$ by player 2, player 1 responds by investing $k_1 = k_2 + \frac{V_2}{s}$. To deter player 1 from investing, player 2 sets the lowest k_2 so that

$$EW_1\left(k_2 + \frac{V_2}{s}, k_2\right) = V_1 - c\left(k_2 + \frac{V_2}{s}\right) \leq 0,$$

which gives a deterrence investment of

$$k_2^D = c^{-1}(V_1) - \frac{V_2}{s}.$$

It is straightforward to show that $k_2^D > 0$ for $s > \bar{s}_2$ as defined in this range. To be profitable for 2, it must be the case that:

$$EW_2(0, k_2^D) = V_2 - c(k_2^D) \geq 0$$

which holds for $s \leq S$. This completes the derivation of the preemptive equilibrium in Case 1, documented in Part 2(a) in Proposition 7.

We now consider Case 2. In this case, $\bar{s}_2 = \frac{V_1+V_2}{c^{-1}(V_1)}$, and for any investment level k_2 by player 2, player 1 responds by investing $k_1 = k_2 + \frac{V_2}{s}$. As shown above, to deter player 1 from investing, player 2 must set $k_2 = k_2^D$, which yields a positive payoff to player 2 if $s \leq S$. Therefore, for all $s \in (\bar{s}_2, S]$, $k_1 = 0, k_2 = k_2^D$ constitutes an equilibrium with preemptive investment. This completes the derivation of the equilibrium strategy in Case 2, documented in Part 2(b) in Proposition 7.

Part 3. Finally, we consider $s > S$. In this case, continuing with either k_2^d or k_2^D will give player 2 a negative expected payoff, so he sets $k_2^{**} = 0$, and player 1 follows with his unique best response $k_1^{**} = \frac{V_2}{s}$. Player 1 gets a strictly positive payoff for $V_1 - c(V_2/s) > 0 \iff s > \frac{V_2}{c^{-1}(V_1)}$, which is certainly the case since $s > \bar{s}_2 > \frac{V_2}{c^{-1}(V_1)}$.

□

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