Endogenous entry and investment in an all-pay auction

Derek J. Clark* Tapas Kundu†

May 8, 2025

Abstract

Contests are ubiquitous but do not simply arise in a vacuum. Competitors make conscious decisions before the fighting stage. This paper looks at the interplay between the decision to enter and then undertake a pre-contest investment to enhance the chance of winning the prize. We use an all-pay auction to model the contest stage; investment cost is private information and its return is stochastic. We characterize equilibrium in terms of threshold strategies on the cost parameter, both for the entry and the subsequent investment decision. We show that the stochastic nature of the investment outcome has a non-monotonic effect on players decisions in equilibrium. A contest designer can use our results to directly achieve goals related to entry fee revenue and investment propensity. In both cases, we demonstrate that limiting entry may be optimal.

JEL Codes: D02, D72, D81, D82

Keywords: All-pay contest; Investment; Endogenous entry; Threshold strategy

*School of Business and Economics, UiT - The Arctic University of Norway. Email: derek.clark@uit.no.

†Corresponding author. Oslo Business School, Oslo Metropolitan University. Email: tapas.kundu@oslomet.no.

1 Introduction

A contest captures strategic interaction between rivals for a prize in which the costs of actions to win are sunk. Examples come from many areas such as political science (e.g. lobbying, armed conflict), economics (e.g. promotions or advertising campaigns) and sports.[1] Much literature assumes that a contest arises among a set of defined competitors with certain characteristics. This paper examines the interplay between factors that may affect actions in the contest prior to the fighting stage. First, we open for the possibility that the set of participants is not known at the beginning of the competition. Development of a novel technology for example often opens new market opportunities, taking time before the set of participants is established. Second, entrants may be able to undertake a pre-contest action that potentially improves the probability of winning and/or the size of the prize. Third, the return to any such action may be uncertain.

We develop a tractable three-period model, involving entry, pre-contest investment and then the actual contest. At the beginning, a fixed set of players are privately informed of their marginal cost of making the pre-contest investment, and must decide whether or not to enter the competition. Next, upon entry, they decide whether to invest in acquiring an advantage which would create a favorable imbalance in the future contest. The return to investment in advantage acquisition is uncertain. Finally, the set of entrants, some of which have potentially acquired an advantage over their rivals, compete in an all-pay auction to win the prize.

By analyzing a model that combines entry and pre-contest investment decisions, we unite disparate strands of the contest literature. The interaction of these decisions appears to have been little studied previously. Investment may enhance the size of a prize that can be appropriated as in Konrad (2002)[2], or reduce the cost of competing for the prize (Fu and Lu, 2009 and Münster, 2007). Entry can occur exogenously as the result of a pre-determined stochastic process (see among others Myerson and Wärneryd, 2006, Münster, 2006, Lim and Matros, 2009 and Fu et al., 2011), or can be set endogenously as part of the equilibrium strategy (for example Fu and Lu, 2010, Fu et al., 2015, Liu and Lu, 2019, Jiao et al., 2022 and Kaplan and Sela, 2010). Some work has been done on the disclosure or concealment of the set of entrants (Jiao et al., 2022), or whether the investment decision is observable (taken simultaneously or sequentially as in Münster, 2007). We combine the approaches by considering endogenous, observable entry and simultaneous investment decisions, the outcome of which is stochastic and observable.

Furthermore we explore how entry is affected by its cost. Kaplan and Sela (2010) and Liu and Lu (2019) study an all-pay auction with an entry cost. Ability is common knowledge and higher ability gives a reduced entry cost and effort cost in the former, while

[1] See Konrad (2009) for an overview of contest types and applications.
[2] Konrad (2002) only considers pre-contest investment by the incumbent.

it is private information in the latter but does not affect the cost of entry. In common with our approach, these papers derive a threshold strategy for entry into the contest. Kaplan and Sela (2010) demonstrate that the contest may not be effective in the sense that the probability of entry is not increasing in ability. To rectify this, they show that the winner of the contest can be charged a fee, although this reduces the attractiveness of entry for all competitors. Liu and Lu (2019) focus attention on the division of the prize mass into different sizes, showing that a single winner-take-all prize is optimal if the cost of effort is linear or concave. The first stage of the model is similar to the contest entry game in Hammond et al. (2019), who analyze an all-pay auction between players with different costs of effort that are private information. Assuming that the entry fee augments the prize, they derive an elegant solution for the entry threshold and total effort. In addition to deriving a threshold for entry, our analysis introduces further uncertainty by considering which of the entrants that will make an ability-enhancing investment before the actual contest is played.

Our findings demonstrate the complex interplay between entry and investment. In making the entry decision, an agent is enticed by the potential value of the contest prize, and upon entry weighs up the private cost of investment with its return. Even if an investment is successful, actions at the contest stage may dissipate its return. The exception is if an agent is the only one to succeed in its investment. Intuitively, contests are hard fought between a sufficiently homogeneous group of competitors, but a single strong player can dissuade rivals from making effort. This forms the incentive to enter and to invest, and we show how this is critically determined by the relationship between the number of agents and the probability of successful investment. Entry and investment are least attractive when the probability of investment success is very high or low, since this imparts a low expectation of being the single strong player (with a successful investment) at the contest stage. Agents are more likely to enter and invest if the success probability takes an intermediate value.

We show also that the entry cost acts as a mechanism to exclude less efficient agents from entering the contest. A designer can effectively use the entry fee to exclude high-cost agents, even though cost information is not freely available.[3] Furthermore, we show how the entry cost may be set in order to achieve a vested interest that the designer may have such us maximizing entry revenue or the expected number of investing agents.

The rest of the paper is organized as follows. The basic model framework is presented in Section 2, and Section 3 sets up the contest. Pre-contest investment decisions are analyzed in Section 4, while Section 5 considers entry strategy. The significance of the model parameters for the analysis is expounded in Section 6. Section 7 considers the

[3]In a similar vein, Fu et al. (2015) show that an effort-maximizing contest designer may wish to limit participation in a Tullock contest with homogeneous participants with endogenous (and concealed), costly entry.

achievement of different goals by a contest designer, and Section 8 concludes. All proofs are in Appendix A.

2 Model

Consider $N \geq 2$ agents who can enter an all-pay auction by incurring an entry cost $c > 0$. Denote the set of entrants by E, which has $n \leq N$ members. After entering, agents can invest in acquiring an advantage that affects their valuation of the contest prize. In this model, there are two equivalent ways of modeling how a successful investment affects payoffs; one can either assume that successful agents have a lower marginal cost of exerting effort in the contest, or that such an agent has a larger prize value.[4] We shall follow the latter interpretation, so that a successful agent has a value of winning the contest of αv where $\alpha > 1$ is common knowledge; the valuation of an unsuccessful or non-investing agent is v. Agent $i \in E$ has an investment cost of θ_i, where $\theta_i, i \in E$ are independent random variables uniformly distributed over $[0, 1]$. The return to investment is stochastic. The likelihood of success is $q \in (0, 1)$, which is the same for every investor and is common knowledge. Those who do not invest do not acquire the advantage. We denote by $m \leq n$ the number of agents that realize a successful return.

The agents that enter the contest exert efforts, given by $\mathbf{x} = (x_1, x_2, ..., x_n)$, in order to win the contest prize. In an all-pay auction, the winner is the contestant with the highest effort; if several agents have the same maximal effort, they each have an equal probability of winning. Let $W(\mathbf{x}) = \{j \in E \mid x_j \geq x_z \text{ for every } z \in E\}$ represent the set of agents that have maximal effort. The probability of agent $i \in E$ winning the contest is given by

$$p_i(\mathbf{x}) = \begin{cases} \frac{1}{|W(\mathbf{x})|} & \text{if } i \in W(\mathbf{x}) \\ 0 & \text{otherwise.} \end{cases}$$

The expected payoff of agent $i \in E$ with prize $V \in \{v, \alpha v\}$ is

$$\pi_i = p_i(\mathbf{x})V - x_i. \tag{1}$$

The game proceeds follows:

- Stage 0: Nature chooses the type θ_i of agent $i = 1, 2, \ldots, N$. The type of an agent is private information.
- Stage 1 (Entry): Agent $i \in \{1, 2, \ldots, N\}$ decides whether to enter, after paying the entry fee. The subset of entering agents is E with $|E| = n$ as the number of

[4] See Vojnović (2015). In an early paper, Konrad (2002) considered an investment made by a single agent – the incumbent – that increased the size of the prize that this agent and the single rival could fight over. Our model permits investment by all agents, and its return accrues as a private benefit.

entrants; n is public information. If $n = 0$, the game ends.

- Stage 2 (Investment): Agent $i \in E$ decides whether to invest by incurring an investment cost of θ_i. Then, nature decides whether an agent realizes a successful investment. The number of agents that realize a successful return, $m \leq n$, is public information.
- Stage 3 (Contest): If $n = 1$, the sole entrant wins the prize. If $n > 1$, agents participate in an all-pay auction to win the prize.

We study the perfect Bayesian equilibrium of the game in symmetric strategies.[5]

3 Contest stage

We begin our analysis at stage 3, where the number of entrants n and the number of successful agents m are common knowledge. Denote the expected contest-stage payoff of a successful agent by $\pi_s(n, m)$, and an unsuccessful one by $\pi_u(n, m)$.[6] Let $T(n, m)$ be the total expected effort exerted in the contest.

If $n = 1$, there is no contest and the sole entrant wins the prize. The entrant's payoff is αv if its investment has paid off, and v if it did not invest, or if the investment failed. We can therefore set $\pi_s(1, 1) = \alpha v$, $\pi_u(1, 0) = v$, and $T(1, 1) = T(1, 0) = 0$.

Consider $n \geq 2$ and $m \in \{0, 1, \ldots, n\}$. Each agent chooses effort to maximize (1). This is then a standard all-pay auction under complete information, which has been extensively studied by Baye et al. (1996). We can use their results directly. There are three cases to consider:[7]

Lemma 1. *(Baye et al., 1996) Suppose that $n \geq 2$.*

(i) $m = 0$. Then $\pi_u(n, 0) = 0$, $T(n, 0) = v$.

(ii) $n \geq m \geq 2$. Then $\pi_s(n, m) = \pi_u(n, m) = 0$, $T(n, m) = \alpha v$.

(iii) $m = 1$. Then $\pi_s(n, 1) = (\alpha - 1) v$, $\pi_u(n, 1) = 0$, and $T(n, 1) \in [T^{min}(n, 1), T^{max}(n, 1)]$

---

[5] Asymmetric equilibria can arise in endogenous–entry contests, where some potential entrants may or may not enter regardless of circumstances, while others follow a threshold-based entry strategy. We analyze the symmetric equilibrium, as it serves as a natural focal point in the absence of a clear coordination mechanism, given that investment type is private information and all firms face identical entry costs. Symmetric equilibria have also been the primary focus of the theoretical literature on contests with endogenous entry; see Fu et al. (2015).

[6] At this stage, an agent may be "unsuccessful" because it did not invest or because it did invest but not succeed. Whatever the source of this lack of success, any investment cost is sunk and this does not affect payoffs at the contest stage.

[7] Cases (i) and (ii) use Theorem 1 in Baye et al. (1996), and case (iii) uses Theorem 2.

where

$$T^{min}(n,1) = \frac{v}{n}\left[(n-1)^2\alpha^2 - (2n-1)(n-1)\alpha + n^2 - (n-1)^2(\alpha-1)^{\frac{2n-1}{n-1}}\alpha^{\frac{-1}{n-1}}\right], \tag{2}$$

$$T^{max}(n,1) = \frac{\alpha+1}{2\alpha}v. \tag{3}$$

In case (i), no agent has realized a successful investment, and all expect a payoff of zero since they exert an expected amount of effort in aggregate that equals the value of the prize. In case (ii), there are at least two successful agents and these exert efforts that are expected to equal their prize value αv. Agents that have not acquired the investment advantage do not exert effort, and all participants expect a payoff of zero. In both of these cases, there are a continuum of equilibria, but Baye et al. (1996) show that they all lead to the same expected total effort. A single successful agent - as in case (iii) - will have a positive expected payoff equal to the difference in the prize between it and a rival with that has not acquired the investment advantage. All unsuccessful agents expect a payoff of zero. Again, there are a continuum of equilibria also in this case, but they do not lead to the same amount of aggregate expected effort. $T(n,1)$ is minimized when the unsuccessful agents all compete in the contest, using a symmetric strategy; this leads to expected effort $T^{min}(n,1)$. On the other hand, Baye et al. (1996) show that $T(n,1)$ is maximized when all but one of the unsuccessful agents have an effort of zero, yielding an expected effort of $T^{max}(n,1)$. When $n=2$, these equilibria coincide since both imply that one successful agent competes with one unsuccessful rival, and $T(2,1) = T^{min}(2,1) = T^{max}(n,1)$.

Because $\alpha > 1 > \frac{\alpha+1}{2\alpha}$, the principal always gets the lowest effort when there is exactly one successful agent, and highest when there is more than one.

4 Investment

Consider stage 2, where the number of entrants n is common knowledge. The investment strategy affects the number of successful players. An agent's investment decision is contingent on his type θ. If there are $m-1 \in \{0,1,\ldots,n-1\}$ successful agents among the other $n-1$ rivals, the return to investment for an agent of type θ is $q\triangle(n,m) - \theta$, where

$$\triangle(n,m) = \pi_s(n,m) - \pi_u(n,m-1).$$

An agent's expected return to investment is $q\mathbb{E}_{m-1}(\triangle(n,m)) - \theta$, where the expectation is taken over the probability distribution of $(m-1)$, the number of successful players among $(n-1)$ competitors.

As investment success is a binary event in our model, the number of successful players follows a Binomial distribution. Specifically, $(m-1) \sim Binomial\,(n-1,\kappa)$ where κ is the probability of finding a successful agent conditional upon entry. This probability κ depends on the investment and entry strategies of the players.

Proposition 1 below shows that every agent's optimal investment strategy is a threshold strategy whenever all agents follow a threshold entry strategy. The intuition is straightforward − conditional upon entry, every agent's expected payoff is decreasing in its own investment cost, and therefore investment pays off only if the cost is sufficiently low. In this case, $\kappa = q\theta_I/\theta_E$, where $\theta_I \in [0,\theta_E]$ and $\theta_E \in [0,1]$ denote the investment and entry thresholds respectively.

Given θ_E, we can derive the investment threshold from the investment-indifference condition:

$$q\left[\sum_{m-1=0}^{n-1}\binom{n-1}{m-1}\left(\frac{q\theta_I}{\theta_E}\right)^{m-1}\left(1-\frac{q\theta_I}{\theta_E}\right)^{n-m}\triangle\,(n,m)\right]-\theta_I=0. \tag{4}$$

Among the $(n-1)$ rivals that the indifferent agent competes against, $(m-1)$ successful agents can be drawn in $\binom{n-1}{m-1}$ ways with the probability of each draw being $\left({}^{q\theta_I}/_{\theta_E}\right)^{m-1}\left(1-{}^{q\theta_I}/_{\theta_E}\right)^{n-m}$ and the indifferent agent becomes the m-th successful agent with probability q after incurring the investment cost θ_I.

Using Lemma 1, we find that

$$\triangle\,(n,m)=\begin{cases}(\alpha-1)\,v & \text{if } m=1\\ 0 & \text{if } m\geq 2\end{cases},$$

which reduces (4) to

$$(\alpha-1)\,vq\left(1-\frac{q\theta_I}{\theta_E}\right)^{n-1}-\theta_I=0. \tag{5}$$

The first term of (5) is the probability of a single agent being successful in a pool of n entrants multiplied by the increment to the agent's payoff in this case; it is easily verified to be strictly decreasing and strictly convex in θ_I. The second term is the threshold investment cost. There will be two possibilities. The left-hand-side of (5) is positive for all $\theta \leq \theta_E$; in this case, every agent with $\theta \leq \theta_E$ invests, and so we can set the investment threshold $\theta_I = \theta_E$. Otherwise, the indifference condition (5) will have a unique solution $\theta_I \leq \theta_E$, which determines the investment threshold. Thus, the following possibilities can arise in equilibrium.

- **Full investment**: For given θ_E and n, all entrants invest: $\theta_I = \theta_E$.
- **Limited investment**: For given θ_E and n, a subset of entrants invest: $\theta_I < \theta_E$.

The condition for full investment will be determined by the marginal entrant's expected return to investment. Denote the gross expected return to investment (without subtracting the investment cost) of an agent under full investment in an $n-$player contest by $\xi(n)$ where

$$\xi(n) := (\alpha - 1) vq (1-q)^{n-1}. \tag{6}$$

It is straightforward to verify that $\xi(n)$ is strictly decreasing in n. The following proposition documents the equilibrium investment strategy for given θ_E and n.

Proposition 1. *Fix θ_E, n, and suppose that all agents with type $\theta \leq \theta_E \leq 1$ enter. There exists $0 < \theta_I \leq \theta_E$ such that all agents with type $\theta \leq \theta_I$ invest in equilibrium.*

1. If $\theta_E \leq \xi(n)$, then $\theta_I = \theta_E$ so that there is full investment.

2. If $\theta_E > \xi(n)$, then there is limited investment and the investment threshold θ_I uniquely solves (5). The investment threshold θ_I weakly increases in θ_E, v and α. Further, (θ_I/θ_E) strictly decreases in θ_E.

Whether all entrants invest or not depends critically on the value of $\xi(n)$. For $n \geq 2$ entrants, it follows from (6) that $\xi(n)$ is the gross expected return to investment under full investment. The regime of full investment is most likely when $\xi(n)$ is high, i.e. when α (the return to investment) or v (the contest prize) are high, and the number of entrants (n) is low. The effect of the probability of successful investment (q) is ambiguous, since $\xi(n)$ is concave in this parameter, increasing for $q \in (0, 1/n)$, and decreasing thereafter. Relatively low or high values of the success probability depends also on the number of entrants since $\xi(n)$ is maximized at $q = 1/n$. When q is relatively low, it is unlikely that an investing agent will succeed, but at the same time if it does succeed, it is likely to be alone; in this case this agent expects $\pi_s(n, 1) = (\alpha - 1) v$ at the contest stage by Proposition 1. A relatively high value of q makes it more likely than an investing agent will succeed, but decreases the chances of being the sole successful agent at the contest stage. This decreases the gross expected return of the investment. This nicely demonstrates the interplay between the entry decision and the success probability in determining whether all entrants invest or not.

Limited investment is most likely to occur if the expected return to investment at the contest stage are low, i.e. low α and/or v, a high number of entrants and a probability of investment success that deviates greatly from $q = 1/n$. The gross expected return to investment under limited investment is θ_I, and this is weakly increasing in the prize parameters associated with the contest stage (α, v).

Recall that c is the cost of entry. For given θ_E and n, let $\pi(\theta, n)$ denote the expected payoff of an agent of type θ at the investment stage, given by:

(a) In case of limited investment, i.e., when $\theta_E > \xi(n)$:

$$\pi(\theta, n) = \begin{cases} \theta_I - \theta - c & \text{if } \theta \leq \theta_I < \theta_E \\ -c & \text{if } \theta_I < \theta \leq \theta_E \\ 0 & \text{if } \theta > \theta_E \end{cases}. \tag{7}$$

(b) In case of full investment, i.e., when $\theta_E \leq \xi(n)$:

$$\pi(\theta, n) = \begin{cases} \xi(n) - \theta - c & \text{if } \theta \leq \theta_I = \theta_E \\ 0 & \text{if } \theta > \theta_E \end{cases}. \tag{8}$$

For $n = 1$, the sole entrant invests if $\theta \leq qv(\alpha - 1) = \xi(1)$, and its expected payoff is $v + \xi(1) - \theta - c$ if it invests, and is $v - c$ if it doesn't invest. Therefore,

$$\pi(\theta, 1) = \begin{cases} v + \xi(1) - \theta - c & \text{if } \theta \leq \min\{\theta_E, \xi(1)\} \\ v - c & \text{if } \min\{\theta_E, \xi(1)\} < \theta \leq \theta_E \\ 0 & \text{if } \theta > \theta_E \end{cases}. \tag{9}$$

For $n = 0$, $\pi(\theta, 0)$ is set to zero.

5 Entry

Consider stage 1. An agent's entry strategy is contingent on its type, which is private information. From (7), (8), and (9), it follows that when there is at least one entrant ($n \geq 1$), the expected payoff of the marginal entrant of type θ_E is

$$\pi(\theta_E, n) = \begin{cases} \begin{cases} \xi(n) - \theta_E - c & \text{if } \theta_E \leq \xi(n) \\ -c & \text{if } \theta_E > \xi(n) \end{cases} & \text{if } n \geq 2 \\ \\ \begin{cases} v + \xi(1) - \theta_E - c & \text{if } \theta_E \leq \xi(1) \\ v - c & \text{if } \theta_E > \xi(1) \end{cases} & \text{if } n = 1 \end{cases}. \tag{10}$$

For the marginal entrant who is indifferent between entry and no entry, the following must hold:

$$\mathbb{E}_{n-1}\left[\pi(\theta_E, n)\right] = \sum_{n-1=0}^{N-1} \binom{N-1}{n-1} (\theta_E)^{n-1} (1 - \theta_E)^{N-n} \pi(\theta_E, n) = 0, \tag{11}$$

where $n - 1 \backsim \text{Binomial}(N - 1, \theta_E)$. The expression in the entry-indifference condition (11) is derived as follows. Among the $(N - 1)$ rivals that the entry-indifferent agent with type θ_E competes against, $n-1 \in \{0,1,\ldots,N-1\}$ other entrants can be drawn in $\binom{N-1}{n-1}$ ways and the probability of each draw is $(\theta_E)^{n-1}(1-\theta_E)^{N-n}$. In each of these draws, the entry-indifferent agent receives an expected payoff of $\pi(\theta_E, n)$ after entering.

The following proposition shows that every agent will adopt a threshold strategy and we formally prove that $\mathbb{E}_{n-1}[\pi(\theta_E, n)]$ is decreasing in θ_E. This observation implies that two possibilities may arise. First, $\mathbb{E}_{n-1}[\pi(\theta_E, n)]$ is always positive for all $\theta \leq 1$; in this case, all types enter, and we can set the entry threshold $\theta_E = 1$. Second, the entry-indifference condition (11) has a unique solution at $\theta_E < 1$, which determines the entry threshold. Thus, we observe the following two possible regimes in equilibrium:

- **Full entry**: All types of agents enter: $\theta_E = 1$.
- **Limited entry**: A subset of agents enters: $\theta_E < 1$.

The expected payoff of the agent of type $\theta = 1$ determines the condition for full entry. Observe that

$$\begin{aligned}\mathbb{E}_{n-1}[\pi(1,n)] = \pi(1,N) &= \begin{cases}\xi(N) - 1 - c & \text{if } 1 \leq \xi(N)\\ -c & \text{if } 1 > \xi(N)\end{cases}\\ &= \underline{c}(N) - c, \end{aligned} \tag{12}$$

where $\underline{c}(N) := \max\{\xi(N) - 1, 0\}$ is the expected gross return of the agent with type $\theta = 1$ under full investment and full entry. The following proposition formally characterizes the equilibrium entry strategy.

Proposition 2. *Fix N. There exists* $0 < \theta_E \leq 1$ such that all agents with type $\theta \leq \theta_E$ enter.

1. If $c \leq \underline{c}(N)$, then $\theta_E = 1$ so that there is full entry.

2. If $c > \underline{c}(N)$, then there is limited entry and the entry threshold θ_E uniquely solves (11). Further, θ_E increases in v and α.

From the entry and investment thresholds, we can fully characterize the distribution of the number of entrants and the number of agents who succeed with an investment. Specifically, n follows *Binomial* (N, θ_E) and m follows *Binomial* $(n, q\theta_I/\theta_E)$.

Full entry is most likely to occur when $\xi(N)$ is large and/or the entry cost (c) is small; the former occurs when the rewards from the contest stage (α, v) are large and the probability of investment success is at an intermediate value (i.e. close to $q = {}^1\!/\!_N$). A straightforward implication of Proposition 2 is that for a contest with an entry fee, full investment occurs whenever there is full entry. This follows from the following observations. First, if $0 < c \leq \underline{c}(N)$, then $\underline{c}(N) > 0$ and hence $n = N$. Further, $\underline{c}(N) > 0$ implies that $\theta_E \leq 1 < \xi(N)$ and by Proposition 1, we get $\theta_I = \theta_E = 1$, summed up in Corollary 1:

Corollary 1. *If* $0 < c \leq \underline{c}(N)$*, then* $\theta_I = \theta_E = 1$*.*

The intuition is as follows. If the full entry condition is satisfied, the marginal entrant faces no uncertainty about the number of entrants. She will always compete against the remaining $N - 1$ players in an all-pay auction. Since a player that does not realize a successful investment receives zero payoff in the all-pay auction, the marginal entrant can never recover its entry cost by not investing. Therefore, in the full-entry regime, the marginal entrant, and consequently every entrant, is committed to invest.

Limited entry implies $n \leq N$, occurring when $c > \underline{c}(N)$, and the investment threshold depends on the realized value of n. The marginal entrant faces uncertainty regarding the number of entrants at the entry stage. If it faces no competition upon entry, which happens when it is the sole entrant, it expects to receive a positive payoff even if it has not realized a successful investment. After entering, the entry cost becomes a sunk cost, and the agent will only invest if the potential gain from investing is sufficient to recover the investment cost. Limited investment can be observed alongside limited entry if $\xi(n)$ falls below the entry threshold θ_E for any given n. Because $\xi(n)$ is decreasing in n, an agent's incentive to invest is reduced with the number of entrants.

The interplay between the uncertainty that arises from entry and that from investment is complex. For a contest with an entry fee, full entry implies full investment, and that is a clean result. Limited entry can give rise to both full investment and limited investment in equilibrium. We explore this further in the next section.

6 Comparative statics

In this section, we discuss the comparative statics effects of some key parameters of our model on entry and investment incentives.[8]

6.1 Likelihood of successful investment

The likelihood of success q has contrasting effects on incentives for investment and entry. When q is high, an agent is more likely to succeed in investment, which favorably affects the payoff from entry. However, this agent also anticipates competing against more agents that have made a successful investment, which has a dampening effect. Typically, both entry and investment incentives are high at an intermediate range of q, and this range is concentrated around $q = 1/N$.

To see why, let us examine the full-entry condition: $c \leq \underline{c}(N)$. Observe that $\xi(N)$ is concave for $q \in [0, 1]$, is equal to zero at $q = \{0, 1\}$, and it increases with q for $q < 1/N$,

[8] It is customary in contest models to focus attention on expected total effort. From Lemma (1) effort is αv, in all cases except $m = 0$ (giving effort v) and $m = 1$ in which case it is $\left\{T^{min}, T^{max}\right\}$. Computing the probabilities of the latter events requires closed-form solutions for the entry and investment thresholds, and these we cannot calculate. Hence we focus on entry and investment incentives.

but decreases thereafter. Therefore, the full-entry condition can only be satisfied at an intermediate level of q. We can hence find an interval $\left[\underline{q}, \overline{q}\right], 0 < \underline{q} \leq \overline{q} < 1$ such that $\left[\underline{q}, \overline{q}\right] = \{q \in [0,1] : c \leq \underline{c}(N)\}$. Further, this interval can be vacuous if $\max_{q \in [0,1]} \xi(N) < 1 + c$, which holds if $(N-1)^{N-1}/N^N < (1+c)/v(\alpha-1)$. We state this formally:

Proposition 3. *Fix $N \geq 2$.*

1. If $(N-1)^{N-1}/N^N < (1+c)/v(\alpha-1)$, there is limited entry in equilibrium for every $q \in [0,1]$.

2. If $(N-1)^{N-1}/N^N \geq (1+c)/v(\alpha-1)$, there exist $0 < \underline{q} \leq 1/N \leq \overline{q} < 1$ such that for $q \in \left[\underline{q}, \overline{q}\right]$, there is full entry in equilibrium.

Part 2 indicates that when the contest is sufficiently favorable (high potential prize value, a low entry cost and few potential competitors), all agents will enter if the success probability balances the positive effect of achieving the advantage with the negative one of meeting potentially strong rivals. When there is full entry, there is also full investment. When the full-entry condition does not hold, the full-investment condition is given by $\theta_E \leq \xi(n)$, where n is the realized number of entrants. For given θ_E and n, it easily follows from the shape of $\xi(n)$ that the full-investment condition, if satisfied, only occurs at an intermediate level of q.

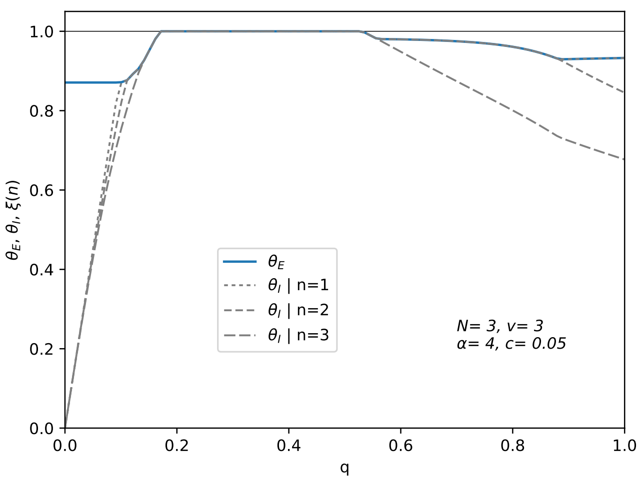

Figure 1: θ_E and θ_I against q for $N = 3$

Part 1 shows that when the contest is expected to be less favorable, then no value of the investment success parameter will entice all agents to enter. This is intuitively straightforward. However, it is less obvious how the thresholds θ_E and θ_I change in relation to q in regimes with limited entry and limited investment.

Figures 1 and 2 illustrate how the two thresholds move against q. In Figure 1, which plots θ_E and θ_I against q for $N = 3$, there is full entry in equilibrium when $q \in [0.169, 0.531]$. Figure 2 plots the threshold for $N = 5$ and there is limited entry

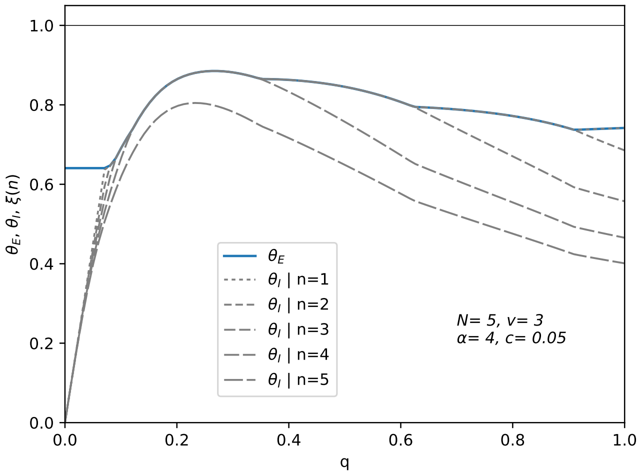

Figure 2: θ_E and θ_I against q for $N = 5$

for every q. These figures also plot θ_I, contingent on $n \in \{1, 2, \ldots, N\}$. The investment threshold θ_I depends on the realized number of entrants. For $q \in [0.169, 0.531]$, $\xi(3) > 1 + c$, and we have full entry in equilibrium. However, since $\xi(5) < 1 + c$, for $N = 5$, there is limited entry in equilibrium for every q.

6.2 Number of agents

The entry incentive weakly diminishes as the total number of agents N increases; this follows from two observations. Firstly, the full-entry condition is satisfied for sufficiently small values of N. Additionally, in cases of limited entry, the entry threshold decreases as N increases. The following proposition documents formally how the entry threshold changes with respect to N.

Proposition 4. *Define* $\bar{N} := \max\{0, 1 + \lfloor(\ln(1+c) - \ln((\alpha-1)vq))/\ln(1-q)\rfloor\}$, *where* $\lfloor x \rfloor$ *is the largest integer less than or equal to* x.

1. *For* $N \leq \bar{N}$, $\theta_E(N) = 1$.
2. *For* $N > \bar{N}$, $\theta_E(N)$ *weakly decreases in* N.

Part 1 is the familiar full-entry condition. Since $N \geq 2$, full entry cannot occur if $\bar{N} < 2$, which happens if then entry cost is very large: $1 + c > \xi(2)$; in this case, two agents will not both find it profitable to enter, and full entry will certainly not occur for additional agents. Entry is less attractive the more agents there are. To get the intuition behind the result, consider from the perspective of the marginal entrant when there are N players. The marginal entrant expects a positive payoff in two scenarios. First, it might be the only entrant, and the expected post-entry payoff in this event is $v + \max\{0, \xi(1) - \theta_E\}$. Second, there could be $(n-1)$ other entrants for various values of n, and the marginal entrant's expected post-entry payoffs in these events are $\max\{0, \xi(n) - \theta_E\}$. When the number of players increases by 1, the likelihood of the first scenario decreases, and the payoffs associated with the second scenario decrease for every n. Consequently, the marginal entrant's expected post-entry payoff declines as N increases.

Describing the impact of N on the investment threshold is more complex because the threshold depends on the realized number of entrants, and the distribution of the number of entrants changes as N moves. If we fix the number of entrants at a given n and examine how changing N affects the investment threshold, we can infer from Proposition 1 that θ_I will also decrease. This is because the two thresholds are positively related.

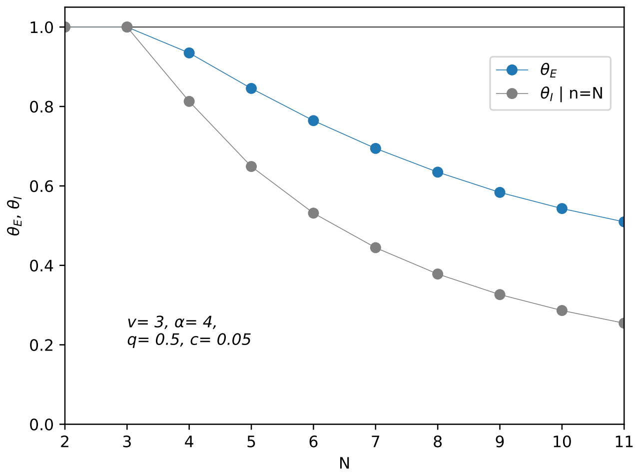

Figure 3: θ_E and $\theta_I\ |_{n=N}$ against N

Figure 3 numerically illustrates the relationship between the entry threshold (shown as the blue-colored curve) and N. We also plot the investment threshold (shown as the grey-colored curve) when all agents have entered ($n = N$); in this case the investment threshold solves:

$$(\alpha - 1)\, vq \left(1 - \frac{q\theta_I}{\theta_E(N)}\right)^{N-1} - \theta_I = 0. \tag{13}$$

As N increases, it affects θ_I in two ways: first, by directly influencing the threshold that solves (13), and second, by decreasing the entry threshold. Proposition 1 implies that the second effect leads to a decline of the investment threshold. Furthermore, it can be shown that the direct effect of N decreases θ_I.[9]

[9]The proof of this follows the technique used to show part 2 of Proposition 4, and is omitted here.

6.3 Entry fee

An entry fee adversely affects the entry incentive. As with our analysis of the effects of N, we can illustrate the dampening effect of c with two observations.

Firstly, the full-entry condition is satisfied only for sufficiently small values of c, specifically, for $c \leq \underline{c}(N)$. In addition, when there is limited entry, we can examine how θ_E moves with respect to c by analyzing (11). Proposition 5 documents the effect of c on θ_E.

Proposition 5. *For $c \leq \underline{c}(N)$, $\theta_E = 1$. For $c > \underline{c}(N)$, θ_E is strictly decreasing in c.*

The mechanism behind this result is straightforward: entry fees directly reduce the marginal entrant's payoff in all possible scenarios, thereby dampening the incentive to enter.

The monotone relationship between the entry fee and the entry threshold has important implications for design problems. A contest designer can achieve her desired entry threshold by adjusting the entry fee. For a given θ_E, let $\widehat{c}(\theta_E)$ be the maximum level of entry cost that results in an entry threshold equal to θ_E. Then we can prove the following proposition.

Proposition 6. *Any entry threshold $\theta_E \in [0,1]$ can be implemented by choosing an entry fee $c = \widehat{c}(\theta_E)$. Furthermore, $\widehat{c}(\theta_E)$ is continuous and differentiable.*

7 Contest design

Contest models often purport the existence of a designer that sets various instruments in the competition in order to achieve some objective. Given Proposition 6, an entry fee may be one such instrument. Entry fees are sometimes charged in order to recoup the expenses from running the contest, or to limit participation, especially by low-quality agents. Taylor (1995) notes that the US Federal Communications Commission opened a contest to design the technology standard for HD-TV, charging an entry fee of 200,000 USD. Competitions in music, writing, sports and architecture often charge an entry fee. In 2023 the participation fees for the Eurovision song contest totaled 6.2 million Euros.[10]

We examine two distinct objectives pursued by the designer. In the first, the designer maximizes the expected total entry fees received. In the second, the designer maximizes the expected number of investors. The analyses below primarily focus on finding conditions in which the designer prefers limited entry.

7.1 Total fees collected

The contest designer can extract surplus from the participants by charging an entry fee, c, to maximize the expected value of the total fees received. As the number of entrants

[10]See Eurovision (2024).

n follows *Binomial* (N, θ_E), the expected value of fees received is $c\mathbb{E}(n) = cN\theta_E(c)$ for a given c. The optimal choice of c, therefore, maximizes $Nc\theta_E(c)$.

By replacing c by $\widehat{c}(\theta_E)$, we can rewrite the optimization problem as a choice problem over the possible entry threshold values. The designer's preferred choice of θ_E maximizes the expected value of fees collected, denoted by V_f:

$$V_f(\theta_E) := N\widehat{c}(\theta_E)\,\theta_E. \tag{14}$$

Because of continuity and differentiability of $\widehat{c}(\theta_E)$, V_f is continuous and differentiable in θ_E. Therefore, if the optimization problem has an interior solution, this must satisfy the first-order necessary condition:

$$\widehat{c}(\theta_E) + \theta_E \frac{d\widehat{c}(\theta_E)}{d\theta_E} = 0.$$

In general, the objective function (14) can exhibit both concave and convex properties. Proposition 7 outlines the sufficient condition under which limited entry is preferred.

Proposition 7. *Consider a contest designer who maximizes the total fees received. If* $\underline{c}(N) = 0$*, then the designer prefers limited entry. If* $\underline{c}(N) > 0$*, then a sufficient condition for the designer to prefer limited entry is given by*

$$\xi(N-1)(1-Nq) - 2 - v \cdot \mathbf{1}_{\{N=2\}} < 0, \tag{15}$$

where $\mathbf{1}_{\{N=2\}}$ *is an indicator function that takes the value* 1 *if* $N = 2$*, and* 0 *otherwise.*

Limited entry is preferred for large N values. This is because $\underline{c}(N) = 0$ for sufficiently high N values. Further, when $\underline{c}(N) > 0$, the sufficient condition (15) is more likely to hold for high values of N: $\xi(N-1)(1-Nq) \leq 0$ for $N \geq 1/q$ and is positive but decreasing in N for $N < 1/q$. Similarly, for sufficiently large q values, limited entry is preferred. Although $\underline{c}(N)$ moves non-monotonically with respect to q, it is decreasing in q for $q \geq 1/N$, and the sufficient condition is always negative for $q \geq 1/N$. It is important to note that if $dV_f/d\theta_E > 0$ as θ_E approaches 1, we cannot definitely conclude that full entry is preferred, as there could be a local interior maximum even if V_f is increasing at $\theta_E = 1$. This is illustrated in Figure 4.

Figure 4 depicts how V_f changes with respect to θ_E under various scenarios. We set $N = 3$, $v = 8$, $\alpha = 7$, and vary q within the set $\{0.08, 0.15, 0.5\}$. In all these scenarios, $\underline{c}(N) > 0$. The sufficiency condition in (15) is met when $q = 0.5$, but it is not satisfied for $q = 0.08$ and $q = 0.15$. For $q = 0.5$, V_f (represented by the continuous curve) reaches its maximum value of 22.59 at $\theta_E = 0.54$. For $q = 0.15$, V_f (represented by the green dot-dashed curve) achieves its maximum at the boundary $\theta_E = 1$. For $q = 0.08$, V_f (shown as the blue dashed curve) attains its interior maximum at $\theta_E = 0.52$.

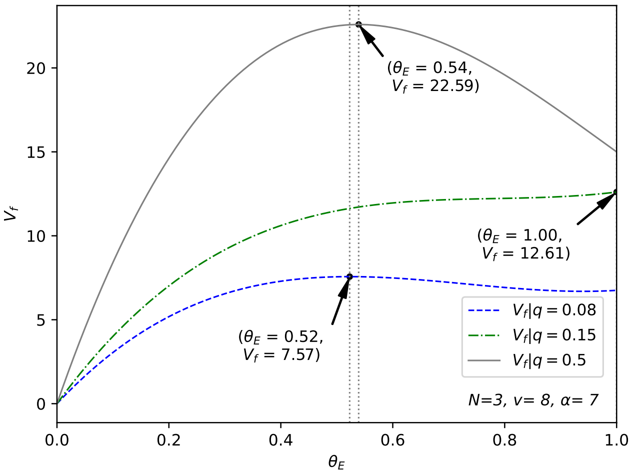

Figure 4: V_f against θ_E for different q values

7.2 The expected number of investors

Consider that the contest designer's objective is to maximize the expected number of investors. The likelihood of investment by a player, conditional upon entry, is θ_I/θ_E. As the number of entrants follows $\text{Binomial}\,(N, \theta_E)$, the expected number of investors, denoted by V_{inv}, is given by

$$V_{inv}\left(\theta_E\right) = \sum_{n=0}^{N} \binom{N}{n} \left(\theta_E\right)^n \left(1-\theta_E\right)^{N-n} \frac{n\theta_I\left(n\right)}{\theta_E}, \tag{16}$$

where $\theta_I\left(n\right)$ and θ_E satisfy the conditions described in Proposition 1 and Proposition 2. After rearranging terms, (16) can be simplified as

$$V_{inv}\left(\theta_E\right) = N\mathbb{E}_{n-1}\left[\theta_I\left(n\right)\right], \tag{17}$$

where $(n-1)$ follows $\text{Binomial}\,(N-1, \theta_E)$.

Consider first the case $\underline{c}\left(N\right) > 0$, which occurs when $\xi\left(N\right) > 1$. Note that if $\theta_E \leq \xi\left(N\right)$, then $\theta_E \leq \xi\left(n\right)$ for every $n \leq N$, and consequently, $\theta_I\left(n\right) = \theta_E$. From (17), $V_{inv}\left(\theta_E\right) = N\theta_E$, which is maximized at $\theta_E = 1$. Therefore, the designer prefers full entry.

Next, consider the case when $\underline{c}\left(N\right) = 0$, which occurs when $\xi\left(N\right) \leq 1$. As we have argued in the previous case, replacing c by $\widehat{c}\left(\theta_E\right)$ in (17), we can express the designer's problem as a choice problem over the possible values of θ_E. Thus, we can study the derivatives of V_{inv} with respect to θ_E at the boundary values to derive a sufficient condition for the existence of a preferred entry threshold strictly below 1. Proposition 8 documents the sufficient condition under which limited entry is preferred.

Proposition 8. *Consider a contest designer who maximizes the expected number of investors. If* $\underline{c}(N) > 0$*, then the designer prefers full entry. If* $\underline{c}(N) = 0$*, then a sufficient condition for the designer to prefer limited entry is given by*

$$\frac{(N-1)\, q\hat{\theta}^2}{(N-2)\, q\hat{\theta}+1} + (N-1)\left(\hat{\theta} - \hat{\hat{\theta}}\right) < 0, \tag{18}$$

where $\hat{\theta} := \lim_{\theta_E \to 1} \theta_I(N)$ *and* $\hat{\hat{\theta}} := \lim_{\theta_E \to 1} \theta_I(N-1)$*.*

Why might a designer choose to limit entry? In scenarios where full investment occurs (i.e., when $\theta_I = \theta_E$), the designer generally benefits from raising the entry threshold. However, she might consider limiting entry specifically when the investment threshold is significantly lower than the entry threshold for certain values of n. In such cases, the designer's motivation for increasing θ_E is influenced not just by the investment thresholds across different events with varying n values, but also by the rate at which the probabilities of these events shift. Notably, as θ_E approaches 1, the rate of change in probabilities of all events, except when $n = N$ and $n = N-1$, asymptotically approaches zero. In contrast, the probabilities of the events of $n = N-1$ and $n = N$ decrease and increase, respectively, as θ_E nears 1. Additionally, given that $\theta_I(N-1)$ is strictly larger than $\theta_I(N)$ (when both are below θ_E), the reduction in probability of $n = N-1$ can sometimes weaken the designer's motivation to raise the entry threshold. The sufficient condition outlined in (18) precisely characterizes such scenarios.

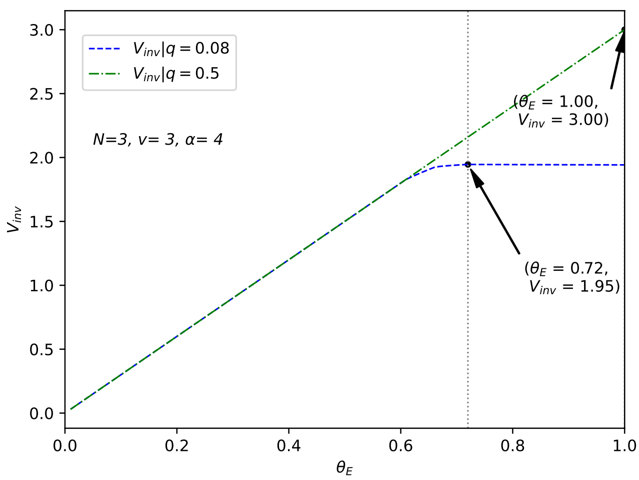

Figure 5: V_{inv} against θ_E for different q values

Figure 5 illustrates how V_{inv} changes with θ_E. We set $N = 3$, $v = 3$, $\alpha = 4$, and consider cases where q is 0.08 and 0.5. For $q = 0.5$, $\underline{c}(N) = 0.12$, and V_{inv} (depicted by the green dot-dashed curve) achieves its maximum at the boundary $\theta_E = 1$. For $q = 0.08$, $\underline{c}(N) = 0$ and the sufficiency condition in (18) is satisfied; V_{inv} (represented by the blue dashed curve) reaches its maximum at $\theta_E = 0.72$, with the maximum value being 1.95. Notably, for $q = 0.08$, V_{inv} at $\theta_E = 1$ is marginally lower, measured at 1.94.

8 Conclusion

This paper has investigated a generic contest model with endogenous entry and investment. Our analysis shows a complex interplay between the two types of uncertainty, captured by the relationship between the probability of success and the total potential number of participants. While the potential size of the prize and the cost of entering the contest have predictable effects on entry and investment, the probability of a successful investment has a non-monotonic effect. Furthermore, the region of non-monotonicity is inextricably linked to the total number of competitors. The threshold values of marginal cost that determine entry and investment in equilibrium resolve the complex decision making process. Agents know that if they meet homogeneous rivals in the final contest (whether they are all successful or unsuccessful in their investment), they will compete away a large proportion of the contested prize, lowering the incentive to enter and make a pre-contest investment. Meeting one or more stronger rivals in the contest also leads to a low expected payoff and weak incentives. The driving force behind entry/investment is the promise of being the lone strong agent in the contest, who is guaranteed a large prize for low effort. If the probability of achieving a successful investment is very low, then an agent is likely to meet equally weak rivals in the upcoming contest. If the success probability is high, an agent will expect to meet several equally strong rivals. An intermediate probability of success balances these two scenarios, making entry and investment more attractive.

We have shown that any entry threshold can be implemented by appropriate setting of the cost of entering the contest. A sufficiently low (but positive) fee can entice full entry into the contest, and this in turns guarantees that all entrants invest. A higher entry cost discourages entry by those who have the highest marginal cost of investment. It may well still be the case that all entrants invest also in this scenario. All other things equal, a large initial number of competitors weakens the incentive for both entry and investment.

Appendix A

Appendix A contains the proofs that are omitted in the main text. We will begin with documentation of two additional results, Lemma A.1 and Lemma A.2, that will be useful in proving our main findings.

Lemma A.1. *Consider a function* $f(p, X) : [0,1] \times \mathbb{N} \to \mathbb{R}$ *that is decreasing in both arguments,* p *and* X*. Let* $m \geq 2$ *be an integer and let* X *follow Binomial*(m, p)*. Then,* $d\mathbb{E}_X [f(p, X)] / dp \leq 0$.
Furthermore, if f *is strictly decreasing in* X *for some* $X \in \{0, 1, ..., m\}$*, or if* f *is strictly decreasing in* p *at some* $X \in \{0, 1, ..., m\}$*, then* $d\mathbb{E}_X [f(p, X)] / dp < 0$.

Proof of Lemma A.1. Assume $X \backsim \text{Binomial}(m, p)$. Then,

$$\mathbb{E}_X [f(p, X)] = \sum_{j=0}^{m} f(p, j) \binom{m}{j} (p)^j (1-p)^{m-j} . \tag{A.1}$$

Claim 1: $d\mathbb{E}_X [f(p, X)] / dp = m\mathbb{E}_Y [f(p, Y+1) - f(p, Y) + (pf_p(p, Y+1)) / Y + 1] + (1-p)^m f_p(p, 0)$, where $Y \backsim \text{Binomial}(m-1, p)$ and $f_p(p, X) = \partial f(p, X) / \partial p$, which is the partial derivative of f with respect p.

Proof of Claim 1: Differentiating (A.1) with respect to p, we get

$$\begin{aligned}
\frac{d}{dp}\mathbb{E}_X [f(p, X)] =& \sum_{j=0}^{m} f(p, j) \binom{m}{j} \left[jp^{j-1} (1-p)^{m-j} - (m-j) p^j (1-p)^{m-j-1} \right] \\
&+ \sum_{j=0}^{m} f_p(p, j) \binom{m}{j} p^j (1-p)^{m-j} \\
=& \sum_{j=1}^{m} f(p, j) j \binom{m}{j} p^{j-1} (1-p)^{m-j} \\
&- \sum_{j=0}^{m-1} f(p, j) (m-j) \binom{m}{j} p^j (1-p)^{m-j-1} \\
&+ \sum_{j=1}^{m} f_p(p, j) \binom{m}{j} p^j (1-p)^{m-j} + (1-p)^m f_p(p, 0)
\end{aligned}$$

Replacing $j\binom{m}{j}$, $(m-j)\binom{m}{j}$, and $\binom{m}{j}$ by $m\binom{m-1}{j-1}$, $m\binom{m-1}{j}$, and $\frac{m}{j}\binom{m-1}{j-1}$, respectively, we get

$$\begin{aligned}\frac{d}{dp}\mathbb{E}_X\left[f\left(p,X\right)\right]=&\sum_{j=1}^{m}f\left(p,j\right)m\binom{m-1}{j-1}p^{j-1}\left(1-p\right)^{(m-1)-(j-1)}\\&-\sum_{j=0}^{m-1}f\left(p,j\right)m\binom{m-1}{j}p^{j}\left(1-p\right)^{m-j-1}\\&+\sum_{j=1}^{m}pf_p\left(p,j\right)\frac{m}{j}\binom{m-1}{j-1}p^{j-1}\left(1-p\right)^{(m-1)-(j-1)}+\left(1-p\right)^{m}f_p\left(p,0\right)\end{aligned}$$

Replacing $j-1$ by j in the first and the third terms,

$$\begin{aligned}\frac{d}{dp}\mathbb{E}_X\left[f\left(p,X\right)\right]=&\sum_{j=0}^{m-1}f\left(p,j+1\right)m\binom{m-1}{j}p^{j}\left(1-p\right)^{(m-1)-j}\\&-\sum_{j=0}^{m-1}f\left(p,j\right)m\binom{m-1}{j}p^{j}\left(1-p\right)^{m-j-1}\\&+\sum_{j=0}^{m-1}pf_p\left(p,j+1\right)\frac{m}{j+1}\binom{m-1}{j}p^{j}\left(1-p\right)^{(m-1)-j}+\left(1-p\right)^{m}f_p\left(p,0\right)\\=&m\mathbb{E}_Y\left[f\left(p,Y+1\right)-f\left(p,Y\right)+\frac{pf_p\left(p,Y+1\right)}{Y+1}\right]+\left(1-p\right)^{m}f_p\left(p,0\right),\end{aligned}\tag{A.2}$$

where $Y\backsim\text{Binomial}\left(m-1,p\right)$. This proves claim 1.

Observe that f is decreasing in both arguments, we have $f\left(p,Y+1\right)\leq f\left(p,Y\right)$ and $f_p\left(p,Y\right)\leq 0$ for any Y. Therefore, $d\mathbb{E}_X\left[f\left(p,X\right)\right]/dp\leq 0$.

Further, it follows from (A.2) that if f is strictly decreasing in Y for some $Y\in\{0,1,\ldots,m-1\}$ or if $f_p<0$ at some $Y\in\{0,1,\ldots,m-1\}$, then $d\mathbb{E}_X\left[f\left(p,X\right)\right]/dp$ is strictly negative, which proves the final part of the Lemma. □

Lemma A.2. *Consider a function $f(X):\mathbb{N}\to\mathbb{R}$ that is decreasing in X. Fix $p\in[0,1]$ and define a function $F:\mathbb{N}\to\mathbb{R}$ by $F\left(m\right)=\mathbb{E}_X\left[f\left(X\right)\right]$ where $X\backsim\text{Binomial}\left(m,p\right)$. Then, $F\left(m+1\right)\leq F\left(m\right)$. Furthermore, the inequality holds strictly if f is strictly decreasing for some $X\in\{0,1,...,m\}$.*

Proof of Lemma A.2. Since $X\backsim\text{Binomial}\left(m,p\right)$, it can be expressed as the sum of m Bernoulli variables: $X=X_1+\ldots+X_m$ where $X_i\backsim\text{Bernoulli}\left(p\right)$. Then, $F\left(m\right)=\mathbb{E}_X\left[f\left(X\right)\right]=\mathbb{E}_{X_1}\cdots\mathbb{E}_{X_m}\left[f\left(X_1+\ldots+X_m\right)\right]$ for any m. Further, given that X_{m+1} follows Bernoulli, we can write

$$\begin{aligned}
F(m+1) =& \mathbb{E}_{X_1} \cdots \mathbb{E}_{X_{m+1}} \left[f(X_1 + \ldots + X_{m+1}) \right] \\
=& \mathbb{E}_{X_1} \cdots \mathbb{E}_{X_m} \left[pf(X_1 + \ldots + X_m + 1) + (1-p) f(X_1 + \ldots + X_m) \right] \\
=& \mathbb{E}_{X_1} \cdots \mathbb{E}_{X_m} \left[p \left(f(X_1 + \ldots + X_m + 1) - f(X_1 + \ldots + X_m) \right) \right] \\
& + \mathbb{E}_{X_1} \cdots \mathbb{E}_{X_m} \left[f(X_1 + \ldots + X_m) \right] \\
=& p\mathbb{E}_X \left[(f(X+1) - f(X)) \right] + F(m), \qquad \text{(A.3)}
\end{aligned}$$

where $X \backsim \text{Binomial}(m, p)$. Because $f(X)$ is decreasing in X, it follows that $F(m+1) \leq F(m)$.

Finally, if $f(X+1) < f(X)$ for some $X \in \{0, 1, \ldots, m\}$, then it follows from (A.3) that $F(m+1) < F(m)$. □

Proof of Lemma 1. Parts (i) and (ii) follow directly from Baye et al. (1996, Theorem 1). Part (iii) uses their Theorem 2. Denoting the expected effort of the skilled agent by e_s, we can use Baye et al. (1996, Theorem 2C) to write the expected sum of efforts as

$$T(n, 1) = \sum_{i=1}^{n} \mathbb{E}x_i = \frac{v}{\alpha} + \left(1 - \frac{1}{\alpha}\right) \mathbb{E}x_s, \qquad \text{(A.4)}$$

where $\mathbb{E}x_s$ is the expected effort of the single skilled agent, and this varies across the continuum of equilibria. Denoting the mixed strategy of the skilled agent by $G_s(x_s), x_s \in \left[\underline{x_s}, \overline{x_s}\right]$, we have

$$\mathbb{E}x_s = \int_{\underline{x_s}}^{\overline{x_s}} (1 - G_s(x_s) dx_s. \qquad \text{(A.5)}$$

In the equilibrium leading to the least effort, we use Baye et al. (1996, eq. 4) to find the mixed strategy of the skilled agent as

$$G_s(x_s) = \frac{x_s}{v} \left(1 - \frac{1}{\alpha} + \frac{x_s}{\alpha v}\right)^{\frac{2-n}{n-1}}, x_s \in [0, v]. \qquad \text{(A.6)}$$

Inserting (A.6) into (A.5) and then into (A.4) gives $T^{min}(n, 1)$ after some rearrangement.

Further, when only one unskilled agent is active, Baye et al. (1996, eq. 4) implies

$$G_s(x_s) = \frac{x_s}{v}, x_s \in [0, v]. \qquad \text{(A.7)}$$

Inserting (A.7) into (A.5) and into (A.4) gives $\mathrm{T}^{max}(n, 1)$. It is straightforward to verify by substitution that $T^{min}(2, 1) = T^{max}(n, 1)$. □

Proof of Proposition 1. Consider that agents are following a threshold entry strategy: all types less than θ_E enter. An agent's return to investment $q\triangle(n, m) - \theta$ is decreasing in its investment cost θ, implying that its investment strategy follows a cutoff rule as well. Further, because all agents have the same entry cost c, the investment cutoff will also be the same across all agents who enter. Denoting the investment threshold by θ_I, the probability that a randomly picked agent would have a successful investment conditional on entry is $q \Pr[\theta \leq \theta_I] / \Pr[\theta \leq \theta_E] = q\theta_I/\theta_E$. Since agents' success are independent events, the probability that an agent faces exactly $m-1$ successful agents out of $n-1$ entrants is given by $\binom{n-1}{m-1}(q\theta_I/\theta_E)^{m-1}(1-(q\theta_I/\theta_E))^{n-m}$, $m-1 \in \{0,\dots,n-1\}$. Therefore, the expected return to investment is

$$q\left[\sum_{m-1=0}^{n-1}\binom{n-1}{m-1}\left(\frac{q\theta_I}{\theta_E}\right)^{m-1}\left(1-\frac{q\theta_I}{\theta_E}\right)^{n-m}\triangle(n,m)\right]-\theta,$$

which further reduces to $(\alpha-1)vq(1-(q\theta_I/\theta_E))^{n-1}-\theta$ because $\triangle(n,m)=0$ for all $m \geq 2$. If the expected return for the marginal entrant is positive, which happens if $(\alpha-1)vq(1-q)^{n-1}-\theta_E \geq 0$, or equivalently, $\xi(n) \geq \theta_E$, then all agents who enter must invest. In this case, $\theta_I = \theta_E$. If the expected return is negative for the agent with type θ_E, which happens if $\xi(n) < \theta_E$, then only a subset of agents must invest, and θ_I uniquely satisfies (5). The uniqueness follows from the fact that the marginal investor's expected return is also decreasing in θ_I.

Let $\mathit{\Omega} := q(1-(q\theta_I/\theta_E))^{n-1}v(\alpha-1)$. From (5) we can find

$$\begin{aligned}\frac{d\theta_I}{d\theta_E} &= \frac{\frac{\partial\mathit{\Omega}}{\partial\theta_E}}{1-\frac{\partial\mathit{\Omega}}{\partial\theta_I}}\\ &= \frac{q\theta_I(n-1)\mathit{\Omega}}{\theta_E\left(\theta_E-q\theta_I+q(n-1)\mathit{\Omega}\right)} > 0.\end{aligned}$$

The positive marginal effects of v and α can also be derived similarly. Furthermore,

$$\begin{aligned}\frac{\partial}{\partial\theta_E}\left(\frac{\theta_I}{\theta_E}\right) &= \frac{\theta_E\frac{d\theta_I}{d\theta_E}-\theta_I}{\theta_E^2}\\ &= \frac{-\theta_I(\theta_E-q\theta_I)}{\theta_E^2\left(\theta_E-q\theta_I+q(n-1)\mathit{\Omega}\right)} < 0.\end{aligned}$$

□

Proof of Proposition 2. Consider the entry decisions of two agents with types θ_1 and θ_2, where $\theta_1 < \theta_2$. The θ_1-type agent can achieve a payoff as high as that of the θ_2-type agent simply by replicating the strategy followed by the θ_2-type agent, and even higher if the strategy involves investment in subsequent subgames. Therefore, the expected payoff of the θ_1-type agent from its optimal entry strategy is greater than that of the θ_2-type agent, for any given strategy profile followed by other players. This observation implies that an agent would adopt a cutoff strategy: enter if and only if θ is below a certain threshold. Furthermore, since all agents face the same entry cost, the threshold is the same for all of them. We denote this threshold as θ_E.

At the entry stage, the expected payoff of the agent of type θ_E is $\mathbb{E}_{n-1}\left[\pi\left(\theta_E, n\right)\right]$ where the number of other players, $n-1$, is a random variable following a Binomial distribution with parameters N and θ_E. It directly follows from Lemma $A.1$ that $\mathbb{E}_{n-1}\left[\pi\left(\theta_E, n\right)\right]$ is decreasing in θ_E. The full-entry condition can therefore be derived from the expected payoff of the agent of type $\theta = 1$, which is given by $\mathbb{E}_{n-1}\left[\pi\left(1, n\right)\right] = \pi\left(1, N\right) = \underline{c}\left(N\right) - c$. Therefore, if $c \leq \underline{c}\left(N\right)$, every agent has an incentive to enter, and $\theta_E = 1$. On the other hand, if $c > \underline{c}\left(N\right)$, $\pi\left(1, N\right)$ is negative and (11) has a unique solution determining the entry threshold.

Further, considering $\mathbb{E}_{n-1}\left[\pi\left(\theta_E, n\right)\right]$ as a function $G\left(\theta_E, z\right)$ of θ_E and a generic parameter z, we can work with the total derivative of (11) to get

$$\frac{d\theta_E}{dz} = -\frac{\partial G/\partial z}{\partial G/\partial \theta_E}.$$

As $\partial G/\partial \theta_E \leq 0$, $d\theta_E/dz$ has the same sign as $\partial G/\partial z$, whenever both terms are well-defined. Applying this observation and the fact that $\pi\left(\theta_E, n\right)$ is increasing in v and α, we conclude that θ_E increases in v and α. □

Proof of Proposition 3. It follows from Proposition 2 that there is limited entry if $c > \underline{c}\left(N\right)$, or equivalently, if $\xi\left(N\right) < 1 + c$, which holds if

$$f_1\left(q\right) := q\left(1-q\right)^{(N-1)} < \frac{1+c}{v\left(\alpha - 1\right)}.$$

Examining the first derivative, we get that f_1 is increasing in $q \leq 1/N$, and decreasing thereafter, implying

$$\max_{q\in[0,1]} f_1\left(q\right) = \frac{\left(N-1\right)^{N-1}}{N^N}.$$

If $\max_{q\in[0,1]} f_1\left(q\right) < \left(1+c\right)/v\left(\alpha-1\right)$, there is limited entry for every $q \in [0,1]$, which proves part (i) of the proposition. Further, if $\max_{q\in[0,1]} f_1\left(q\right) \geq \left(1+c\right)/v\left(\alpha-1\right)$, then there will be full entry for some q. Given that f_1 is increasing up to $1/N$ and decreasing thereafter, $f_1\left(q\right)$ must be higher than $\left(1+c\right)/v\left(\alpha-1\right)$ at an interval $\left[\underline{q}, \overline{q}\right]$, containing $1/N$. □

Proof of Proposition 4. Observe that the full-entry condition $c \leq \underline{c}\left(N\right)$ can be rewritten as

$$\left(1+c\right) \leq \xi\left(N\right) \Leftrightarrow \left(1-q\right)^{N-1} \geq \frac{\left(1+c\right)}{vq\left(\alpha-1\right)}.$$

By taking the logarithm on both sides and noting that $\ln(1-q)$ is negative, we can express the above inequality as

$$N \leq 1 + \frac{\ln\left(\left(1+c\right)/\left(vq\left(\alpha-1\right)\right)\right)}{\ln\left(1-q\right)}.$$

Defining $\bar{N}$ as $\max\{0, 1 + \lfloor(\ln(1+c) - \ln((\alpha-1)vq))/\ln(1-q)\rfloor\}$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x, part (1) of the proposition directly follows.

Next, suppose $N > \bar{N}$, in which case, there is limited entry and θ_E satisfies (11). We express $\mathbb{E}_{n-1}[\pi(\theta_E, n)]$, where $(n-1)$ follows the distribution Binomial $(N-1, \theta_E)$, as a function of θ_E and N and denoted by $G_1(\theta_E, N)$:

$$G_1(\theta_E, N) = \sum_{n-1=0}^{N-1} \binom{N-1}{n-1} (\theta_E)^{n-1} (1-\theta_E)^{N-n} \pi(\theta_E, n).$$

The entry threshold $\theta_E(N)$ implicitly solves $G_1(\theta_E, N) = 0$. Therefore,

$$\begin{aligned} & G_1(\theta_E(N+1), N+1) - G_1(\theta_E(N), N) = 0 \\ \Leftrightarrow & [G_1(\theta_E(N+1), N+1) - G_1(\theta_E(N+1), N)] \\ & + [G_1(\theta_E(N+1), N) - G_1(\theta_E(N), N)] = 0. \end{aligned}$$

Since $\pi(\theta_E, n)$ is decreasing in n, it follows from Lemma $A.2$ that $G_1(\theta_E, N)$ is decreasing in N, which implies that $G_1(\theta_E(N+1), N+1) \leq G_1(\theta_E(N+1), N)$. Consequently, $G_1(\theta_E(N+1), N) \geq G_1(\theta_E(N), N)$. However, as $\pi(\theta_E, n)$ is also decreasing in θ_E, by applying Lemma $A.1$, we find that $G_1(\theta_E, N)$ decreases in θ_E. Therefore, it must be that $\theta_E(N+1) \leq \theta_E(N)$, which completes the proof. □

Proof of Proposition 5. The first part of the proposition directly follows from Proposition 2. In order to show that θ_E is decreasing in c, we consider $\mathbb{E}_{n-1}[\pi(\theta_E, n)]$, where $(n-1)$ follows the distribution Binomial $(N-1, \theta_E)$, as a function of θ_E and N and denote it by $G_2(\theta_E, c)$. Note that $\theta_E(c)$ implicitly solves $G_2(\theta_E, c) = 0$. Taking the total derivative of G_2 along the path of $\theta_E(c)$, we get $d\theta_E/dc = -(dG_2/dc)/(dG_2/d\theta_E)$.

Note that $\pi(\theta_E, c)$ is strictly decreasing in c, and therefore it follows from Lemma $A.1$ that $\partial G_2/\partial c < 0$. Further, as θ_E solves $\mathbb{E}_{n-1}[\pi(\theta_E, n)] = 0$, it must be that $\theta_E \leq \xi(n)$ at least for some n (as otherwise $\mathbb{E}_{n-1}[\pi(\theta_E, n)]$ will be independent of θ_E), which implies that $\pi(\theta_E, n)$ is strictly decreasing in θ_E for some n. Therefore, by applying Lemma $A.1$, we get $\partial G_2/\partial\theta_E < 0$. Hence, $d\theta_E/dc < 0$, which completes the proof. □

Proof of Proposition 6. To construct $\widehat{c}(\theta_E)$, we consider the two cases separately, $\underline{c}(N) > 0$ and $\underline{c}(N) = 0$. Consider first $\underline{c}(N) > 0$. It follows from Proposition 2 that for all $c \leq \underline{c}(N)$, $\theta_E = 1$, and therefore, $\widehat{c}(1) = \underline{c}(N)$. For $c > \underline{c}(N)$ and $\theta_E < 1$, θ_E and c have a one-to-one relationship satisfying (11). Therefore, for all $\theta_E < 1$, $\widehat{c}(\theta_E)$ is uniquely determined by the solution of (11). Further, by Proposition 5, $\widehat{c}(\theta_E)$ is strictly decreasing for $0 \leq \theta_E \leq 1$. Because $G_2(\theta_E, c)$ is continuous and differentiable in c, $\theta_E(c)$ is also continuous and differentiable in c, ensuring the continuity and differentiability of $\widehat{c}(\theta_E)$ in $\theta_E \in [0,1]$. Further, because of strict monotonicity of $\widehat{c}(\theta_E)$, any entry threshold θ_E in $[0,1]$ can be implemented by choosing an entry fee $c = \widehat{c}(\theta_E)$. Next, consider $\underline{c}(N) = 0$, which occurs when $\xi(N) \leq 1$. We claim that if $c = 0$, then $\theta_E = 1$ is a unique solution of (11). The proof follows from two observations. Firstly, at $\theta_E = 1$, $\mathbb{E}_{n-1}[\pi(1,n)] = \pi(1,N) = 0$. Secondly, with $c = 0$, we have for all $\theta_E < 1$, $\pi(\theta_E, 1) > 0$ and $\pi(\theta_E, n) \geq 0$ for $n \geq 2$. Therefore, $\mathbb{E}_{n-1}[\pi(\theta_E, n)] > 0$ for all $\theta_E < 1$, implying that any $\theta_E < 1$ cannot be a solution of (11) if $c = 0$. For $c > 0$ and $\theta_E < 1$, θ_E and c have a one-to-one relationship satisfying (11), and therefore, $\widehat{c}(\theta_E)$ is uniquely determined by the solution of (11). Further, as we have argued in the previous case, $\widehat{c}(\theta_E)$ is differentiable and strictly decreasing in θ_E for all $\theta_E \in [0,1]$. Therefore, any entry threshold θ_E in $[0,1]$ can be implemented by setting $c = \widehat{c}(\theta_E)$. □

Proof of Proposition 7. It follows from the discussion in Section 6.3 that $\widehat{c}(\theta_E)$ is strictly positive for all $\theta_E < 1$. Therefore, the maximum value of $V_f(\theta_E)$ must be positive, and it must reach its maximum at some $\theta_E > 0$.

Let us first consider the case $\underline{c}(N) = 0$. Then, $\widehat{c}(1) = \underline{c}(N) = 0$, and therefore, $V_f(1) = 0$, which implies that V_f is maximized at some interior $\theta_E \in (0,1)$, and so the designer prefers limited entry.

Next, consider $\underline{c}(N) > 0$. Therefore, $\widehat{c}(1) = \underline{c}(N) = \xi(N) - 1$. We derive a sufficient condition for an interior maximum by examining the derivative of V_f as θ_E approaches 1: if the derivative is negative, then V_f must be maximized at some $0 < \theta_E < 1$. Note that

$$\lim_{\theta_E \to 1} \frac{dV_f}{d\theta_E} = N\left(\widehat{c}(1) + \lim_{\theta_E \to 1} \frac{d\widehat{c}(\theta_E)}{d\theta_E}\right).$$

Recall from the proof of Proposition 5 that for $\theta_E \in (0,1)$ and $c > 0$, $\widehat{c}(\theta_E)$ solves

$$G_2(\theta_E, c) = \sum_{n=1}^{N} \binom{N-1}{n-1} (\theta_E)^{n-1} (1-\theta_E)^{N-n} \pi(\theta_E, n) = 0.$$

From the total differential of $dG_2 = 0$ along the path of $\widehat{c}(\theta_E)$, we can derive $d\widehat{c}(\theta_E)/d\theta_E = -(\partial G_2/\partial\theta_E)/(\partial G_2/\partial c)$. Further,

$$dG_2/dc = \sum_{n=1}^{N} \binom{N-1}{n-1} (\theta_E)^{n-1} (1-\theta_E)^{N-n} \frac{d\pi(\theta_E, n)}{dc} = -1,$$

which gives us $d\widehat{c}(\theta_E)/d\theta_E = (dG_2/d\theta_E)$, and

$$\lim_{\theta_E \to 1} \frac{dV_f}{d\theta_E} = N\left(\xi(N) - 1 + \lim_{\theta_E \to 1} \frac{dG_2(\theta_E, c)}{d\theta_E}\right). \tag{A.8}$$

Differentiating $G_2(\theta_E, c)$ with respect to θ_E, term by term, and taking the limit as $\theta_E \to 1$, we get

$$\lim_{\theta_E \to 1} \frac{dG_2(\theta_E, c)}{d\theta_E} = \left[(N-1)\lim_{\theta_E \to 1} \pi(\theta_E, N) + \lim_{\theta_E \to 1} \frac{d\pi(\theta_E, N)}{d\theta_E}\right] - \left[(N-1)\lim_{\theta_E \to 1} \pi(\theta_E, N-1)\right], \tag{A.9}$$

where the first square-bracketed term arises from the derivative of the last term in the summation series, and the second square-bracketed term comes from the derivative of the second-to-last term of the series; Because $d\pi(\theta_E, n)/d\theta_E$ is finite for any n, it can be easily shown that the derivatives of all other terms approach zero in the limit as θ_E approaches 1.

Note that as $\xi(N) > 1$, from (10), we get $\lim_{\theta_E \to 1} d\pi(\theta_E, N)/d\theta_E = -1$. Further, as $\theta_E \to 1$, $\widehat{c}(\theta_E) \to \widehat{c}(1) = \underline{c}(N) = \xi(N) - 1$. Therefore,

$$\lim_{\theta_E \to 1} \pi(\theta_E, N) = \xi(N) - 1 - \widehat{c}(1) = 0, \text{ and}$$
$$\lim_{\theta_E \to 1} \pi(\theta_E, N-1) = v \cdot \mathbf{1}_{\{N=2\}} + \xi(N-1) - 1 - \widehat{c}(1) = v \cdot \mathbf{1}_{\{N=2\}} + \xi(N-1) - \xi(N)$$

where $\mathbf{1}_{\{N=2\}}$ is an indicator function that takes the value 1 if $N = 2$, and 0 otherwise. Replacing the limiting values in the right-hand-side of (A.9), we get

$$\lim_{\theta_E \to 1} \frac{dG_2(\theta_E, c)}{d\theta_E} = (N-1)\left(\xi(N) - \xi(N-1) - v \cdot \mathbf{1}_{\{N=2\}}\right) - 1.$$

Further, replacing the limiting value of $dG_2(\theta_E, c)/d\theta_E$ in (A.8) and using the fact that $(N-1)v \cdot \mathbf{1}_{\{N=2\}} = v \cdot \mathbf{1}_{\{N=2\}}$, we can express

$$\begin{aligned} \lim_{\theta_E \to 1} \frac{dV_f}{d\theta_E} &= N\left[N\xi(N) - (N-1)\xi(N-1) - 2 - v \cdot \mathbf{1}_{\{N=2\}}\right] \\ &= N\left[\xi(N-1)(1 - Nq) - 2 - v \cdot \mathbf{1}_{\{N=2\}}\right]. \end{aligned}$$

Therefore, (15) implies that $\lim_{\theta_E \to 1} dV_f/d\theta_E < 0$ and it provides a sufficient condition for having an interior maximum. □

Proof of Proposition 8. It follows from (17) that $V_{inv} = 0$ at $\theta_E = 0$, and $V_{inv} > 0$ at $\theta_E = 1$, implying that V_{inv} reaches its maximum at some $\theta_E > 0$.

Let us first consider the case $\underline{c}(N) > 0$, which occurs if $\xi(N) > 1$. In this case, $\theta_E \leq \xi(n)$ for all $\theta_E \in [0, 1]$ and $n \leq N$, and therefore, by Proposition 1, $\theta_I = \theta_E$ and

$V_{inv} = N\theta_E$, which is increasing in θ_E. Hence, the designer prefers full entry.

Next, consider $\underline{c}(N) = 0$. Then, as $\theta_E \to 1$, we have $\widehat{c}(\theta_E) \to \widehat{c}(1) = \underline{c}(N) = 0$. We will derive a sufficient condition for an interior maximum by examining the derivative of V_{inv} as θ_E approaches 1: if the derivative is negative, then V_{inv} must be maximized at some $0 < \theta_E < 1$. Note that

$$\lim_{\theta_E \to 1} \frac{dV_{inv}(\theta_E)}{d\theta_E} = N \lim_{\theta_E \to 1} \frac{d}{d\theta_E} \left[\sum_{n=1}^{N} \binom{N-1}{n-1} (\theta_E)^{n-1} (1-\theta_E)^{N-n} \theta_I(n) \right].$$

Differentiating $V_{inv}(\theta_E)$ with respect to θ_E, term by term, and taking the limit as $\theta_E \to 1$, we get

$$\lim_{\theta_E \to 1} \frac{dV_{inv}(\theta_E)}{d\theta_E} = N \left[(N-1) \lim_{\theta_E \to 1} \theta_I(N) + \lim_{\theta_E \to 1} \frac{d\theta_I(N)}{d\theta_E} \right] - N \left[(N-1) \lim_{\theta_E \to 1} \theta_I(N-1) \right], \tag{A.10}$$

where the first square-bracketed term arises from the derivative of the last term in the summation series, and the second square-bracketed term comes from the derivative of the second-to-last term of the series; Because $d\theta_I(n)/d\theta_E$ is finite for any n, it can be easily shown that the derivatives of all other terms approach zero in the limit as θ_E approaches 1.

To find $\lim_{\theta_E \to 1} d\theta_I(N)/d\theta_E$, observe that $\theta_I(N)$ solves

$$f(\theta_I, \theta_E) := v(\alpha-1) q (1 - (q\theta_I/\theta_E))^{N-1} - \theta_I = 0.$$

Therefore, from the total differential of $df = 0$ along the path of $\theta_I(N)$, we can derive $d\theta_I(N)/d\theta_E = -(\partial f/\partial\theta_E)/(\partial f/\partial\theta_I)$. Further,

$$\begin{aligned} \frac{\partial f}{\partial \theta_E} &= (N-1) v(\alpha-1) q (1-(q\theta_I/\theta_E))^{N-2} \left(\frac{q\theta_I}{\theta_E^2}\right), \text{ and} \\ \frac{\partial f}{\partial \theta_I} &= -(N-1) v(\alpha-1) q (1-(q\theta_I/\theta_E))^{N-2} \left(\frac{q}{\theta_E}\right) - 1, \end{aligned}$$

which give us

$$\lim_{\theta_E \to 1} \frac{d\theta_I(N)}{d\theta_E} = \frac{(N-1) v(\alpha-1) q^2 \left(1-q\hat{\theta}\right)^{N-2} \hat{\theta}}{(N-1) v(\alpha-1) q^2 \left(1-q\hat{\theta}\right)^{N-2} + 1}, \tag{A.11}$$

where $\hat{\theta} := \lim_{\theta_E \to 1} \theta_I(N)$. Because $f(\theta_I, \theta_E)$ is continuous in θ_I and $\theta_E > 0$, $\hat{\theta}$ satisfies $\hat{\theta} = v(\alpha-1) q \left(1-q\hat{\theta}\right)^{N-1}$. Therefore, (A.11) can be simplified as

$$\lim_{\theta_E \to 1} \frac{d\theta_I(N)}{d\theta_E} = \frac{(N-1) q\hat{\theta}^2}{(N-1) q\hat{\theta} + \left(1-q\hat{\theta}\right)} = \frac{(N-1) q\hat{\theta}^2}{(N-2) q\hat{\theta} + 1} \in (0,1). \tag{A.12}$$

We define $\hat{\hat{\theta}} := \lim_{\theta_E \to 1} \theta_I (N-1)$, which solves $v(\alpha-1) q (1-q\theta)^{N-2} - \theta = 0$. It can be easily verified that $\hat{\hat{\theta}} > \hat{\theta}$.

Using (A.12), we can simplify (A.11) as

$$\lim_{\theta_E \to 1} \frac{dV_{inv}(\theta_E)}{d\theta_E} = N \left[(N-1) \left(\hat{\theta} - \hat{\hat{\theta}} \right) + \frac{(N-1) q \hat{\theta}^2}{(N-2) q \hat{\theta} + 1} \right].$$

Therefore, (18) implies that $\lim_{\theta_E \to 1} dV_{inv}/d\theta_E < 0$ and it provides a sufficient condition for having an interior maximum. □

Disclosure statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Author contributions

Derek J. Clark: Conceptualization, Methodology, Formal analysis, Investigation, Writing – original draft. **Tapas Kundu:** Conceptualization, Methodology, Formal analysis, Investigation, Writing – original draft.

Acknowledgements

We thank the seminar participants at the 10th Annual Conference on Contests: Theory and Evidence (University of Reading, UK), the 18th Annual Conference on Economic Growth and Development (Indian Statistical Institute, New Delhi, India), and our colleagues at the School of Business and Economics, UiT, Norway, and Oslo Business School, Oslo Metropolitan University, Norway, for their helpful comments. All remaining errors are our own.

References

Baye, M. R., Kovenock, D., and De Vries, C. G. (1996). The all-pay auction with complete information. *Economic Theory*, 8(2):291–305.

Eurovision (2024). Was kostet der ESC die Teilnehmerländer? `https://www.eurovision.de/news/Was-kostet-der-ESC-die-Teilnehmerlaender,gebuehren110.html#:~:text=Bei%20der%20Finanzierung%20des%20ESC,als%20Teilnahmegeb%C3%BChr%20anteilig%20per%20Umlage`. In German. Accessed 06 May, 2025.

Fu, Q., Jiao, Q., and Lu, J. (2011). On disclosure policy in contests with stochastic entry. *Public Choice*, 148:419–434.

Fu, Q., Jiao, Q., and Lu, J. (2015). Contests with endogenous entry. *International Journal of Game Theory*, 44:387–424.

Fu, Q. and Lu, J. (2009). Contest with pre-contest investment. *Economics Letters*, 103(3):142–145.

Fu, Q. and Lu, J. (2010). Contest design and optimal endogenous entry. *Economic Inquiry*, 48(1):80–88.

Hammond, R. G., Liu, B., Lu, J., and Riyanto, Y. E. (2019). Enhancing effort supply with prize-augmenting entry fees: Theory and experiments. *International Economic Review*, 60(3):1063–1096.

Jiao, Q., Ke, C., and Liu, Y. (2022). When to disclose the number of contestants: Theory and experimental evidence. *Journal of Economic Behavior & Organization*, 193:146–160.

Kaplan, T. R. and Sela, A. (2010). Effective contests. *Economics Letters*, 106(1):38–41.

Konrad, K. A. (2002). Investment in the absence of property rights; the role of incumbency advantages. *European Economic Review*, 46(8):1521–1537.

Konrad, K. A. (2009). *Strategy and Dynamics in Contests*. Oxford University Press, Oxford, UK.

Lim, W. and Matros, A. (2009). Contests with a stochastic number of players. *Games and Economic Behavior*, 67(2):584–597.

Liu, B. and Lu, J. (2019). The optimal allocation of prizes in contests with costly entry. *International Journal of Industrial Organization*, 66:137–161.

Münster, J. (2006). Contests with an unknown number of contestants. *Public Choice*, 129(3-4):353–368.

Münster, J. (2007). Contests with investment. *Managerial and Decision Economics*, 28(8):849–862.

Myerson, R. B. and Wärneryd, K. (2006). Population uncertainty in contests. *Economic Theory*, 27:469–474.

Taylor, C. R. (1995). Digging for golden carrots: An analysis of research tournaments. *The American Economic Review*, pages 872–890.

Vojnović, M. (2015). *Contest theory: Incentive mechanisms and ranking methods*. Cambridge University Press.