Appendix B: Investing to get a head start in contests—Supplementary materials

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This document contains supplementary materials for Clark et al. (2025). In section B.1, we analyze the first-period investment game with a quadratic investment cost function $c(k) = \frac{k^2}{2}$. In section B.2, we discuss an equilibrium refinement.

B.1 The investment game with a quadratic cost

The following are the first-period payoff functions.

$$u_1(k_1, k_2) = \begin{cases} V_1 - \dfrac{{k_1}^2}{2}, & \text{if } k_1 > k_2 + \dfrac{V_2}{s}; \\ (V_1 - V_2) + s(k_1 - k_2) - \dfrac{{k_1}^2}{2}, & \text{if } k_1 \in [k_2 + \dfrac{V_2}{s} - \dfrac{V_1}{s}, k_2 + \dfrac{V_2}{s}]; \\ -\dfrac{{k_1}^2}{2}, & \text{if } k_1 < k_2 + \dfrac{V_2}{s} - \dfrac{V_1}{s}. \end{cases} \tag{B.1}$$

$$u_2(k_1, k_2) = \begin{cases} V_2 - \dfrac{{k_2}^2}{2}, & \text{if } k_2 > k_1 + \dfrac{V_1}{s}; \\ (V_2 - V_1) + s(k_2 - k_1) - \dfrac{{k_2}^2}{2}, & \text{if } k_2 \in [k_1 + \dfrac{V_1}{s} - \dfrac{V_2}{s}, k_1 + \dfrac{V_1}{s}]; \\ -\dfrac{{k_2}^2}{2}, & \text{if } k_2 < k_1 + \dfrac{V_1}{s} - \dfrac{V_2}{s}. \end{cases} \tag{B.2}$$

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B.1.1 Pure-strategy equilibria

Observe that $z(s) = \arg\max_{k \geq 0} sk - \frac{k^2}{2} = s$, $M(s) = \frac{s^2}{2}$, $\hat{s}_1 = \sqrt{V_2}$, $\hat{s}_2 = \sqrt{V_1}$, and $c^{-1}(V_i) = \sqrt{2V_i}$. Furthermore, we have

$$\begin{aligned}
&z(\frac{V_1+V_2}{c^{-1}(V_2)}) > c^{-1}(V_2), \text{ which implies } \bar{s}_1 = \frac{V_1+V_2}{c^{-1}(V_2)} = \frac{V_1+V_2}{\sqrt{2V_2}};\\
&z(\frac{V_1+V_2}{c^{-1}(V_1)}) < c^{-1}(V_1), \text{ which implies } \bar{s}_2 = M^{-1}(V_2) = \sqrt{2V_2}; \text{ and}\\
&M(s) = c(z(s)) = \frac{s^2}{2}, \text{ which implies } \underline{s}_2 = \sqrt{2(V_1 - V_2)}.
\end{aligned}$$

Finally, $M(\hat{s}_2) \leq V_2 \Leftrightarrow V_1 \leq 2V_2$. It therefore follows from Propositions 1, 2, and 3 of Clark et al. (2025) that the reinforcement equilibrium exists for all $0 < s \leq \frac{V_1+V_2}{\sqrt{2V_2}}$, and the preemptive equilibrium exists for $s \in [\sqrt{2(V_1-V_2)}, \sqrt{2V_2}]$ if $V_1 \leq 2V_2$.

B.1.2 Mixed-strategy equilibria

Assume that player i is playing in equilibrium a mixed strategy given by the cumulative distribution function G_i. We let $\alpha_i(x)$ denote the mass placed at x by player i's mixed strategy.

Given G_1 and G_2, we can express the players' expected payoffs as follows:

$$\begin{aligned}
EU_1(k_1) = &\int_0^{\max\{k_1 - \frac{V_2}{s}, 0\}} V_1 dG_2(k_2)\\
&+ \int_{\max\{k_1 - \frac{V_2}{s}, 0\}}^{k_1 - \frac{V_2}{s} + \frac{V_1}{s}} ((V_1 - V_2) + s(k_1 - k_2)) dG_2(k_2) - \frac{k_1^2}{2}; \qquad \text{(B.3)}
\end{aligned}$$

$$\begin{aligned}
EU_2(k_2) = &\int_0^{\max\{k_2 - \frac{V_1}{s}, 0\}} V_2 dG_1(k_1)\\
&+ \int_{\max\{k_2 - \frac{V_1}{s}, 0\}}^{\max\{k_2 - \frac{V_1}{s} + \frac{V_2}{s}, 0\}} ((V_2 - V_1) + s(k_2 - k_1)) dG_1(k_1) - \frac{k_2^2}{2}. \qquad \text{(B.4)}
\end{aligned}$$

Using the Leibniz rule, we can derive the first-order derivatives of the players' expected payoff functions, which are given by

$$\frac{dEU_1}{dk_1} = \begin{cases} s\Big[G_2\big(k_1 - \frac{V_2}{s} + \frac{V_1}{s}\big) - G_2\big(k_1 - \frac{V_2}{s}\big)\Big] - k_1, & \text{if } k_1 \geq \frac{V_2}{s},\\ s\Big[G_2\big(k_1 - \frac{V_2}{s} + \frac{V_1}{s}\big) - G_2(0)\Big] - k_1, & \text{if } k_1 < \frac{V_2}{s}; \end{cases} \qquad \text{(B.5)}$$

$$\frac{dEU_2}{dk_2} = \begin{cases} s\Big[G_1\Big(k_2 - \frac{V_1}{s} + \frac{V_2}{s}\Big) \; - \; G_1\Big(k_2 - \frac{V_1}{s}\Big)\Big] \; - \; k_2, & \text{if } k_2 \; > \; \frac{V_1}{s}, \\ s\Big[G_1\Big(k_2 - \frac{V_1}{s} + \frac{V_2}{s}\Big) \; - \; G_1(0)\Big] \; - \; k_2, & \text{if } \frac{V_1 - V_2}{s} \; \leq \; k_2 \; \leq \; \frac{V_1}{s}, \\ - \, k_2, & \text{if } k_2 \; < \; \frac{V_1 - V_2}{s}. \end{cases} \tag{B.6}$$

Observe that, for any given investment level k, player i's payoff is bounded above by $V_i - \frac{k^2}{2}$, so player i would never use a strategy that puts mass on $(\sqrt{2V_i}, \infty)$ (setting the investment equal to zero strictly dominates such a strategy). Therefore, we must have $G_i(k) = 1$ for all $k \geq \sqrt{2V_i}$.

Lemma B.1. $G_2(0) = G_2(\frac{V_1 - V_2}{s})$.

Proof. It follows from (B.6) that 2's payoff is strictly decreasing in k_2 for $k_2 \in (0, \frac{V_1 - V_2}{s}]$ for any given strategy by player 1. $\square$

A direct implication of Lemma B.1 is $\alpha_2(\frac{V_1 - V_2}{s}) = 0$. Furthermore, it also follows that 2 cannot randomize continuously over an interval starting at $\frac{V_1 - V_2}{s}$.

Lemma B.2. *If player 2 randomizes continuously over* $[\frac{V_1 - V_2}{s} + \delta_1, \frac{V_1 - V_2}{s} + \delta_2]$ *with* $\delta_2 > \delta_1 \geq 0$*, we must have* $\delta_1 \neq 0$*.*

Proof. Suppose, toward a contradiction, 2 randomizes over $[\frac{V_1 - V_2}{s}, \frac{V_1 - V_2}{s} + \delta_2]$ with $\delta_2 > 0$. Then, $\frac{dEU_2}{dk_2}$ must be zero at $k_2 = \frac{V_1 - V_2}{s}$. However, by (B.6), $\frac{dEU_2}{dk_2} = -\frac{V_1 - V_2}{s}$ at $k_2 = \frac{V_1 - V_2}{s}$, which is strictly negative. $\square$

Lemma B.3. *If* $\alpha_2(0) > 0$*, then player* 2 *gets zero expected payoff in equilibrium.*

Proof. For any given G_1, it follows from (B.4) that 2's expected payoff from playing zero is zero, which must equal his expected payoff from playing the mixed strategy if he places a positive mass at zero. $\square$

Lemma B.4. *If* $\alpha_1(0) > 0$*, then player* 1*'s expected payoff in equilibrium is* $(V_1 - V_2)\alpha_2(0)$*.*

Proof. By Lemma B.1, for a given G_2, player 1's expected payoff from playing zero is $(V_1 - V_2)G_2(\frac{V_1 - V_2}{s}) = (V_1 - V_2)G_2(0) = (V_1 - V_2)\alpha_2(0)$, which must equal his expected payoff from playing the mixed strategy if he places a positive mass at zero. $\square$

Lemma B.5. *Consider $\delta_2 > \delta_1 > 0$. If 2 randomizes continuously over $[\frac{V_1-V_2}{s} + \delta_1, \frac{V_1-V_2}{s} + \delta_2]$, then 1 must randomize continuously over $[\delta_1, \delta_2]$, and vice versa.*

Proof. We prove this by contradiction. First, consider the possibility that player 2 randomizes continuously over $[\frac{V_1-V_2}{s} + \delta_1, \frac{V_1-V_2}{s} + \delta_2]$ and player 1 does not over $[\delta_1, \delta_2]$, so that there exist ϵ_1, ϵ_2 with $\delta_1 \leq \epsilon_1 < \epsilon_2 \leq \delta_2$ such that $G_1(\epsilon_2) = G_1(\epsilon_1)$.

Since player 2 randomizes over $[\frac{V_1-V_2}{s}+\delta_1, \frac{V_1-V_2}{s}+\delta_2]$, we must have $dEU_2/dk_2 = 0$ at $k_2 = \frac{V_1-V_2}{s} + \epsilon_1$. We claim that, for an ϵ sufficiently close to but above ϵ_1, dEU_2/dk_2, computed at $k_2 = \frac{V_1-V_2}{s} + \epsilon$, is strictly negative. There are two cases to consider.

Case (i): Consider $\epsilon_1 < \frac{V_2}{s}$, and fix $\epsilon \in (\epsilon_1, \min\{\frac{V_2}{s}, \epsilon_2\})$. By (B.6), at $k_2 = \frac{V_1-V_2}{s} + \epsilon$, $dEU_2/dk_2 = G_1(\epsilon) - G_1(0) - \epsilon - \frac{V_1-V_2}{s}$, which is strictly negative, since $\epsilon > \epsilon_1$, $G_1(\epsilon) = G_1(\epsilon_1)$; and at $k_2 = \frac{V_1-V_2}{s} + \epsilon_1$, $dEU_2/dk_2 = G_1(\epsilon_1) - G_1(0) - \epsilon_1 - \frac{V_1-V_2}{s} = 0$.

Case (ii): Consider $\epsilon_1 \geq \frac{V_2}{s}$, and fix $\epsilon \in (\epsilon_1, \epsilon_2)$. By (B.6), at $k_2 = \frac{V_1-V_2}{s} + \epsilon$, $dEU_2/dk_2 = G_1(\epsilon) - G_1(\frac{V_2}{s} + \epsilon) - \epsilon - \frac{V_1-V_2}{s}$, which is also strictly negative since $\epsilon > \epsilon_1$, $G_1(\epsilon) = G_1(\epsilon_1)$, $G_1(\frac{V_2}{s}+\epsilon) \geq G_1(\frac{V_2}{s}+\epsilon_1)$; and at $k_2 = \frac{V_1-V_2}{s}+\epsilon_1$, $dEU_2/dk_2 = G_1(\epsilon_1) - G_1(\frac{V_2}{s} + \epsilon_1) - \epsilon_1 - \frac{V_1-V_2}{s} = 0$.

Therefore, there exists an open interval starting from ϵ_1 in which player 2 will not randomize, contradicting that G_2 is the best response against G_1. The converse case can be proved using a similar line of argument. □

From Lemma B.4 and the observation that $G_2(k_2) = 1$ for all $k_2 \geq \sqrt{2V_2}$, we conclude that $\frac{V_1-V_2}{s} + \delta_2 \leq \sqrt{2V_2}$, or, equivalently,

$$\delta_2 \leq \bar{\delta_2} := \frac{V_2}{s} + \left(\sqrt{2V_2} - \frac{V_1}{s}\right). \tag{B.7}$$

Therefore, for $s < \frac{V_1}{\sqrt{2V_2}}$, $\bar{\delta_2} < \frac{V_2}{s}$.

Lemma B.6. *Assume player 2 mixes continuously with strictly positive probability over an interval $\left[\frac{V_1-V_2}{s} + \delta_1, \frac{V_1-V_2}{s} + \delta_2\right]$, where $0 \leq \delta_1 < \delta_2$. Then there is no nontrivial subinterval in $\left(\frac{V_1-V_2}{s} + \delta_1, \frac{V_1-V_2}{s} + \delta_2\right)$ on which G_2 remains constant. Equivalently, for any $x \in \left(\frac{V_1-V_2}{s} + \delta_1, \frac{V_1-V_2}{s} + \delta_2\right)$ and any small $\epsilon > 0$ with $x \pm \varepsilon$ still in that open interval, we have*

$$G_2(x + \varepsilon) \;-\; G_2(x - \varepsilon) \;>\; 0.$$

Proof. Suppose, for contradiction, that there is a sub-interval $\big(a, b\big) \subset \left(\frac{V_1-V_2}{s} + \delta_1, \frac{V_1-V_2}{s} + \delta_2\right)$ on which G_2 remains flat, *i.e.*, for all $u, v \in (a, b)$, $G_2(u) = G_2(v)$.

Equivalently, player 2 places no probability mass at any point $k_2 \in (a, b)$. By assumption, player 2 is mixing with positive probability throughout $\left[\frac{V_1-V_2}{s} + \delta_1,\ \frac{V_1-V_2}{s} + \delta_2\right]$ and so must be indifferent (*i.e.*, $\frac{d\,EU_2}{dk_2} = 0$) on every point of the support, including points close to a and b.

Since a and b both lie strictly in the interior, player 2 is indeed mixing on $(\,\frac{V_1-V_2}{s} + \delta_1,\ a\,)$ and also on $(\,b,\ \frac{V_1-V_2}{s} + \delta_2\,)$. Hence, we must have

$$\frac{d\,EU_2}{dk_2}\bigg|_{k_2=a} = 0 \quad \text{and} \quad \frac{d\,EU_2}{dk_2}\bigg|_{k_2=b} = 0,$$

because a and b are endpoints of portions of player 2's mixing support.

From (B.6),

$$\frac{d\,EU_2}{dk_2} = G_1(k_2 - \frac{V_1 - V_2}{s}) - G_1(0) - k_2.$$

Setting $k_2 = a$ yields

$$0 = G_1(a - \frac{V_1 - V_2}{s}) - G_1(0) - a, \tag{B.8}$$

and similarly, for $k_2 = b$,

$$0 = G_1(b - \frac{V_1 - V_2}{s}) - G_1(0) - b. \tag{B.9}$$

Subtracting (B.8) from (B.9) and rewriting, we get

$$G_1\Big(b - \frac{V_1 - V_2}{s}\Big) - G_1\Big(a - \frac{V_1 - V_2}{s}\Big) = b - a. \tag{B.10}$$

Let $x \in (a, b)$. Then

$$\frac{d\,EU_2}{dk_2}\bigg|_{k_2=x} = G_1(x - \frac{V_1 - V_2}{s}) - G_1(0) - x.$$

We need to see if this can remain 0 given that it is 0 at $x = a$ and $x = b$.

Consider (B.10). If G_1 is monotonic and $x \in (a, b)$, then $x - a < b - a$ implies

$$G_1(x - \frac{V_1 - V_2}{s}) - G_1(a - \frac{V_1 - V_2}{s}) < G_1(b - \frac{V_1 - V_2}{s}) - G_1(a - \frac{V_1 - V_2}{s}),$$

so that

$$G_1(x - \frac{V_1 - V_2}{s}) - G_1(a - \frac{V_1 - V_2}{s}) < (b - a).$$

Rearranging gives

$$G_1(x - \frac{V_1 - V_2}{s}) \ - \ G_1(0) \ < \ \Big[G_1(a - \frac{V_1 - V_2}{s}) - G_1(0)\Big] + (b - a).$$

But (B.8) implies

$$G_1(x - \frac{V_1 - V_2}{s}) \ - \ G_1(0) \ < \ a + (b - a) \ = \ b.$$

So,

$$\frac{d\,EU_2}{dk_2}\Big|_{k_2 = x} \ = \ \Big[G_1(x - \frac{V_1 - V_2}{s}) - G_1(0)\Big] \ - \ x \ \leq \ b - x \ - \ x \ = \ b - 2\,x.$$

If x is even slightly larger than $\frac{b}{2}$, then the right side becomes negative, forcing $\frac{d\,EU_2}{dk_2}\Big|_{k_2=x} < 0$. Thus the slope cannot remain zero in (a, b). Therefore, player 2 would want to shift probability either into or out of the subinterval (a, b), contradicting the assumption that 2 is fully indifferent at a and b while placing no mass in between. $\square$

It follows from Lemma B.2 and Lemma B.6 that player 2 can randomize over at most one continuous interval of non-zero length in any mixed-strategy equilibrium, and that this interval must begin at $\frac{V_1 - V_2}{s} + \delta_1$, where $\delta_1 > 0$. Consequently, by Lemma B.5, player 1 can randomize over at most one continuous interval, which must begin at δ_1. Below, we consider mixed strategies by player 1 that involve randomization over actions in a continuous interval $[\delta_1, \delta_2]$ with $\delta_2 \geq \delta_1 > 0$ and mixed strategies by player 2 that involve randomization over actions in $[\frac{V_1 - V_2}{s} + \delta_1, \frac{V_1 - V_2}{s} + \delta_2]$.

If such a strategy profile can be sustained in a mixed-strategy equilibrium, then we must have

$$\frac{d\,EU_1}{dk_1} \begin{cases} = 0 \text{ for all } k_1 \in [\delta_1, \delta_2], \\ \leq 0 \text{ for all } k_1 \notin [\delta_1, \delta_2]; \end{cases} \tag{B.11}$$

$$\frac{d\,EU_2}{dk_2} \begin{cases} = 0 \text{ for all } k_2 \in [\frac{V_1 - V_2}{s} + \delta_1, \frac{V_1 - V_2}{s} + \delta_2], \\ \leq 0 \text{ for all } k_2 \notin [\frac{V_1 - V_2}{s} + \delta_1, \frac{V_1 - V_2}{s} + \delta_2]. \end{cases} \tag{B.12}$$

Derivation of the distribution functions

We first derive the optimal distribution functions that sustain randomization over an interval of actions for each player. Later, we examine when players assign a mass point at zero.

Consider the indifference condition of player 1. To begin with, assume that δ_2 is a sufficiently large number (ignoring that G_is are bounded above by 1).

Consider $k \in [\delta_1, \frac{V_1}{s} + \delta_1)$. In this range, $k - \frac{V_2}{s} < \frac{V_1 - V_2}{s} + \delta_1$, and because player 2 does not randomize between 0 and $\frac{V_1-V_2}{s} + \delta_1$, $G_2(k - \frac{V_2}{s}) = G_2(0) = \alpha_2(0)$. From (B.5) and (B.11), $G_2(k + \frac{V_1-V_2}{s}) = G_2(0) + \frac{k}{s}$, or, equivalently,

$$G_2(x) = \frac{x}{s} + \alpha_2(0) - \frac{V_1 - V_2}{s^2} \text{ for } x \in [\frac{V_1 - V_2}{s} + \delta_1, \frac{2V_1 - V_2}{s} + \delta_1)$$

Since player 2 does not randomize between 0 and $\frac{V_1-V_2}{s} + \delta_1$, there will be a mass point at $\frac{V_1-V_2}{s} + \delta_1$, and $\alpha_2(\frac{V_1-V_2}{s} + \delta_1) = \frac{\delta_1}{s}$.

Next, consider $k \in [\frac{V_1}{s} + \delta_1, \frac{2V_1}{s} + \delta_1)$. In this range, $\frac{V_1-V_2}{s} + \delta_1 \le k - \frac{V_2}{s} < \frac{2V_1-V_2}{s} + \delta_1$. From (B.5) and (B.11), $G_2(k + \frac{V_1-V_2}{s}) = G_2(k - \frac{V_2}{s}) + \frac{k}{s} = \frac{k}{s} - \frac{V_2}{s^2} + \alpha_2(0) - \frac{V_1-V_2}{s^2} + \frac{k}{s} = \frac{2k}{s} - \frac{V_1}{s^2} + \alpha_2(0)$, or, equivalently,

$$\begin{aligned} G_2(x) &= \frac{2}{s}\left(x - \frac{V_1 - V_2}{s}\right) - \frac{V_1}{s^2} + \alpha_2(0) \\ &= \frac{2x}{s} + \alpha_2(0) - \frac{3V_1 - 2V_2}{s^2}, \text{ for } x \in [\frac{2V_1 - V_2}{s} + \delta_1, \frac{3V_1 - V_2}{s} + \delta_1). \end{aligned}$$

Next, consider $k \in [\frac{2V_1}{s} + \delta_1, \frac{3V_1}{s} + \delta_1)$. In this range, $\frac{2V_1-V_2}{s} + \delta_1 \le k - \frac{V_2}{s} < \frac{3V_1-V_2}{s} + \delta_1$. From (B.5) and (B.11), $G_2(k + \frac{V_1-V_2}{s}) = G_2(k - \frac{V_2}{s}) + \frac{k}{s} = \frac{2k}{s} - \frac{2V_2}{s^2} + \alpha_2(0) - \frac{3V_1-2V_2}{s^2} + \frac{k}{s} = \frac{3k}{s} - \frac{3V_1}{s^2} + \alpha_2(0)$, or, equivalently,

$$\begin{aligned} G_2(x) &= \frac{3}{s}\left(x - \frac{V_1 - V_2}{s}\right) - \frac{3V_1}{s^2} + \alpha_2(0) \\ &= \frac{3x}{s} + \alpha_2(0) - \frac{6V_1 - 3V_2}{s^2}, \text{ for } x \in [\frac{3V_1 - V_2}{s} + \delta_1, \frac{4V_1 - V_2}{s} + \delta_1). \end{aligned}$$

In general, we can express the distribution as follows (can be verified by induction):

$$G_2(x) = \begin{cases} \alpha_2(0) & \text{for } x < \dfrac{V_1 - V_2}{s} + \delta_1 \\ \min\left\{ \dfrac{nx}{s} + \alpha_2(0) - \dfrac{n(n+1)V_1 - 2nV_2}{2s^2}, 1 \right\} & \text{for } x \in \left[\dfrac{nV_1 - V_2}{s} + \delta_1, \min\left\{ \dfrac{(n+1)V_1 - V_2}{s} + \delta_1, \sqrt{2V_2} \right\} \right), \\ & n = 1, 2, \ldots \\ 1 & \text{for } x \in \left[\sqrt{2V_2}, \infty \right) \end{cases} \tag{B.13}$$

It is easy to verify that $\frac{dEU_1}{dk_1} < 0$ for $k_1 > x$, whenever $G_2(x + \frac{V_1 - V_2}{s}) = 1$. Therefore, for given δ_1, $\alpha_2(0)$, and G_2, (B.5) is satisfied. Computing G_2 at the boundary points of each interval, we find:

$$G_2\left(\frac{nV_1 - V_2}{s} + \delta_1 \right) = \frac{n\delta_1}{s} + \alpha_2(0) + \frac{n(n-1)V_1}{2s^2}, \tag{B.14}$$

$$\lim_{x \to \left(\frac{(n+1)V_1 - V_2}{s} + \delta_1 \right)^-} G_2(x) = \frac{n\delta_1}{s} + \alpha_2(0) + \frac{n(n+1)V_1}{2s^2}, \tag{B.15}$$

$$G_2\left(\frac{(n+1)V_1 - V_2}{s} + \delta_1 \right) = \frac{(n+1)\delta_1}{s} + \alpha_2(0) + \frac{n(n+1)V_1}{2s^2} \tag{B.16}$$

Therefore, G_2 assigns an atom of size $\frac{\delta_1}{s}$ at every boundary point $\frac{nV_1 - V_2}{s} + \delta_1, n \geq 1$. It is also worth noting that G_2 can be equal to one at some k_2 strictly below $\sqrt{2V_2}$.

We next construct G_1 from player 2's indifference condition. Since player 1 does not randomize between 0 and δ_1, we must have

$$G_1(x) = G_1(0) = \alpha_1(0), \text{ for } x < \delta_1. \tag{B.17}$$

Consider $k \in [\frac{V_1 - V_2}{s} + \delta_1, \frac{V_1}{s} + \delta_1)$. In this range, $k - \frac{V_1}{s} < \delta_1$, and because player 1 does not randomize between 0 and δ_1, $G_1(k - \frac{V_1}{s}) = G_1(0) = \alpha_1(0)$. From (B.6) and (B.12), we have $G_1(k - \frac{V_1 - V_2}{s}) = G_1(k - \frac{V_1}{s}) + \frac{k}{s} = \alpha_1(0) + \frac{k}{s}$, or, equivalently,

$$G_1(x) = \frac{x}{s} + \alpha_1(0) + \frac{V_1 - V_2}{s^2}, \text{ for } x \in [\delta_1, \frac{V_2}{s} + \delta_1).$$

Next, consider $k \in [\frac{V_1}{s} + \delta_1, \frac{V_1 + V_2}{s} + \delta_1)$. In this range, $\delta_1 \leq k - \frac{V_1}{s} < \frac{V_2}{s} + \delta_1$, and so $G_1(k-\frac{V_1}{s})=\frac{k}{s}+\alpha_1(0)-\frac{V_2}{s^2}$. From (B.6) and (B.12), we have $G_1(k-\frac{V_1-V_2}{s})=G_1(k-\frac{V_1}{s})+\frac{k}{s}=\frac{2k}{s}+\alpha_1(0)-\frac{V_2}{s^2}$, or, equivalently,

$$G_1(x)=\frac{2x}{s}+\alpha_1(0)+\frac{2V_1-3V_2}{s^2}, \text{ for } x\in[\frac{V_2}{s}+\delta_1,\frac{2V_2}{s}+\delta_1)$$

Next, consider $k\in[\frac{V_1+V_2}{s}+\delta_1,\frac{V_1+2V_2}{s}+\delta_1)$. In this range, $\frac{V_2}{s}+\delta_1\le k-\frac{V_1}{s}<\frac{2V_2}{s}+\delta_1$, and so $G_1(k-\frac{V_1}{s})=\frac{2k}{s}+\alpha_1(0)-\frac{3V_2}{s^2}$. From (B.6) and (B.12), we have $G_1(k-\frac{V_1-V_2}{s})=G_1(k-\frac{V_1}{s})+\frac{k}{s}=\frac{3k}{s}+\alpha_1(0)-\frac{3V_2}{s^2}$, or, equivalently,

$$G_1(x)=\frac{3x}{s}+\alpha_1(0)+\frac{3V_1-6V_2}{s^2}, \text{ for } x\in[\frac{2V_2}{s}+\delta_1,\frac{3V_2}{s}+\delta_1).$$

In general, we can express the distribution as follows (can be proved by induction):

$$G_1(x)=\begin{cases}\alpha_1(0) & \text{for } x<\delta_1\\ \min\left\{\frac{mx}{s}+\alpha_1(0)+\frac{2mV_1-m(m+1)V_2}{2s^2},1\right\} & \text{for } x\in\left[\frac{(m-1)V_2}{s}+\delta_1,\ \min\left\{\frac{mV_2}{s}+\delta_1,\bar{\delta}_2\right\}\right),\ m=1,2,\ldots\\ 1 & \text{for } x\in\left[\bar{\delta}_2,\infty\right)\end{cases}\tag{B.18}$$

where $\bar{\delta}_2=\frac{V_2}{s}+\left(\sqrt{2V_2}-\frac{V_1}{s}\right)$.

Since player 2 will never use a strategy that puts mass on $\left[\sqrt{2V_2},\infty\right)$, player 1 has no incentive to invest at any level above $\bar{\delta}_2$, as we have shown above. However, $G_1(x)$ can be equal to one at some $x<\bar{\delta}_2$. It is easy to verify that $\frac{dEU_2}{dk_2}<0$ for $k_2>\frac{V_1-V_2}{s}+x$ whenever $G_1(x)=1$. For given δ_1, $\alpha_1(0)$, and G_1, (B.6) is satisfied. Computing G_1 at the boundary points of each interval, we find:

$$G_1\left(\frac{(m-1)V_2}{s}+\delta_1\right)=\frac{m\delta_1}{s}+\alpha_1(0)+\frac{m(2V_1+(m-3)V_2)}{2s^2};\tag{B.19}$$

$$\lim_{x\to(\frac{mV_2}{s}+\delta_1)^-}G_1(x)=\frac{m\delta_1}{s}+\alpha_1(0)+\frac{m(2V_1+(m-1)V_2)}{2s^2};\ \text{and}\tag{B.20}$$

$$G_1\left(\frac{mV_2}{s}+\delta_1\right)=\frac{(m+1)\delta_1}{s}+\alpha_1(0)+\frac{(m+1)(2V_1+(m-2)V_2)}{2s^2}.\tag{B.21}$$

Therefore, G_1 assigns an atom of size $\frac{\delta_1}{s}+\frac{V_1-V_2}{s^2}$ at every boundary point $\frac{(m-1)V_2}{s}+\delta_1, m\geq 1$.

Lemma B.7. $\alpha_i(0)\in(0,1)$ *for* $i=1,2$.

Proof. If player 2 plays according to (B.13), then player 1's payoff from playing 0 is $(V_1-V_2)G_2(0)=(V_1-V_2)\alpha_2(0)$, and 1's payoff from playing δ_1 is $(V_1-V_2+\min\{\delta_1 s,V_2\})G_2(0)-\frac{\delta_1^2}{2}=(V_1-V_2+\min\{\delta_1 s,V_2\})\alpha_2(0)-\frac{\delta_1^2}{2}$ (because for any $k_2\geq\frac{V_1-V_2}{s}+\delta_1$, we have $U_1(\delta_1,k_2)=-\frac{\delta_1^2}{2}$). Therefore, in any mixed strategy equilibrium in which 1 randomizes over $[\delta_1,\delta_2]\bigcup\{0\}$, we must have:

$$\begin{aligned}\min\{\delta_1 s,V_2\}\alpha_2(0)>\frac{\delta_1^2}{2}&\implies\alpha_1(0)=0,\\ \min\{\delta_1 s,V_2\}\alpha_2(0)<\frac{\delta_1^2}{2}&\implies\alpha_1(0)=1,\\ \alpha_1(0)\in(0,1)&\implies\min\{\delta_1 s,V_2\}\alpha_2(0)=\frac{\delta_1^2}{2}.\end{aligned}$$

Now, if $\alpha_2(0)=0$, then player 1 must play a pure strategy (specifically, 0). We know from our analysis of pure-strategy equilibria that player 2 then has a unique best response in pure strategy and so will not randomize over a nonzero interval. This rules out the possibility that $\alpha_2(0)=0$ and $\alpha_1(0)=1$.

Similarly, given that player 1 plays according to (B.19), we have that player 2's respective payoffs from playing zero and playing $(\frac{V_1-V_2}{s}+\delta_1)$ are 0 and $\min\{\delta_1 s,V_2\}\alpha_1(0)-(\frac{V_1-V_2}{s}+\delta_1)^2/2$, respectively. Therefore, if player 2 randomizes over $[\frac{V_1-V_2}{s}+\delta_1,\frac{V_1-V_2}{s}+\delta_2]\bigcup\{0\}$, we must have:

$$\begin{aligned}\min\{\delta_1 s,V_2\}\alpha_1(0)>\frac{(\frac{V_1-V_2}{s}+\delta_1)^2}{2}&\implies\alpha_2(0)=0,\\ \min\{\delta_1 s,V_2\}\alpha_1(0)<\frac{(\frac{V_1-V_2}{s}+\delta_1)^2}{2}&\implies\alpha_2(0)=1,\\ \alpha_2(0)\in(0,1)&\implies\min\{\delta_1 s,V_2\}\alpha_1(0)=\frac{(\frac{V_1-V_2}{s}+\delta_1)^2}{2}.\end{aligned}$$

If $\alpha_1(0)=0$, then player 2 must play a pure strategy (specifically, 0), in which case player 1 has a unique best response in pure strategy, and so he will not randomize over a non-zero interval. This rules out the possibility that $\alpha_1(0)=0$ and $\alpha_2(0)=1$.

Therefore, in any mixed-strategy equilibrium, we can only have $\alpha_i(0)\in(0,1)$ for $i=1,2$. □

Lemma B.8. $\alpha_1(0)=\dfrac{(\frac{V_1-V_2}{s}+\delta_1)^2}{2\min\{\delta_1 s,V_2\}}$, $\alpha_2(0)=\dfrac{\delta_1^2}{2\min\{\delta_1 s,V_2\}}$.

Proof. The proof follows from the proof of Lemma B.7. □

By Lemma B.5 and Lemma B.6, both players randomize over at most one continuous interval of the same length. From the specifications of G_1 and G_2 in (B.19) and (B.13), we see that the continuous interval over player 1's action space is a union of contiguous sub-intervals of length V_2/s, while that over player 2's action space is a union of contiguous sub-intervals of length V_1/s. Furthermore, player 1 assigns higher probability masses to the boundary points of each sub-interval compared to player 2; specifically, player 1 assigns a probability of $\frac{\delta_1}{s} + \frac{V_1 - V_2}{s^2}$, whereas player 2 assigns a probability of $\frac{\delta_1}{s}$. These observations, along with the fact that $\alpha_1(0) > \alpha_2(0)$ (which follows from Lemma B.8), together imply that G_1 increases in value at a faster rate than G_2. We therefore define

$$\delta_2 := \liminf_x \{x : G_1(x) = 1\}, \tag{B.22}$$

and adjust player 2's strategy to ensure randomization over an interval of the same length as player 1's as follows:

$$\tilde{G}_2(k_2) := \begin{cases} G_2(k_2) \text{ if } k_2 < \delta_2 \\ 1 \text{ if } k_2 \geq \delta_2 \end{cases}. \tag{B.23}$$

In the above construction, a family of mixed-strategy equilibria is defined by δ_1, and we can express $\alpha_1(0)$, $\alpha_2(0)$, and δ_2 (when it is strictly below $\bar{\delta_2}$) as functions of δ_1. These mixed-strategy equilibria exist only if $G_1(\delta_1) < 1$; if, on the other hand, $G_1(\delta_1) \geq 1$, then one of the players has incentives to deviate. This existence condition can be expressed as:

$$\begin{aligned} G_1(\delta_1) = \alpha_1(0) + \frac{\delta_1}{s} + \frac{V_1 - V_2}{s^2} < 1 \\ \iff \frac{(\frac{V_1 - V_2}{s} + \delta_1)^2}{2\min\{\delta_1 s, V_2\}} + \frac{\delta_1}{s} + \frac{V_1 - V_2}{s^2} < 1 \end{aligned} \tag{B.24}$$

The left-hand side of the above expression changes with respect to δ_1 in a non-monotone way. It can be shown that the expression is increasing in δ_1 for $\delta_1 > V_2/s$ but can have a local minimum at some $\delta_1 < V_2/s$ for V_2 close to V_1. The expression is decreasing in s and it trivially follows that, for $s \leq \sqrt{V_1 - V_2}$, none of these mixed-strategy equilibria can exist (although it is not a necessary condition for non-existence). Below we show that a mixed-strategy equilibrium exists for so high values of s that no reinforcement equilibrium exists.

Steps to construct mixed-strategy equilibria

Step 1: Consider G_1 in (B.19). Let $\alpha_1(0) = \dfrac{(\frac{V_1 - V_2}{s} + \delta_1)^2}{2\min\{\delta_1 s, V_2\}}$ and find δ_1 that satisfies (B.24).

Step 2: Let $\delta_2 = \liminf_x\{x : G_1(x) = 1\}$ and $\alpha_2(0) = \dfrac{\delta_1^2}{2\min\{\delta_1 s, V_2\}}$. Consider G_2 in (B.13) and adjust player 2's strategy to $\tilde{G}_2$ as given in (B.23).

The strategy profile $\{G_1, \tilde{G}_2\}$ constitutes a mixed-strategy equilibrium. In this mixed-strategy equilibrium, player 1's expected payoff is given by $(V_1 - V_2)\alpha_2(0)$, and player 2's expected payoff is zero.

B.1.3 Payoff comparison across equilibria

It trivially follows that a reinforcement (pure-strategy) equilibrium weakly payoff-dominates a mixed-strategy equilibrium in the range of parameter values where both types of equilibria exist. This is because, in any reinforcement equilibrium, player 1 receives at least $V_1 - V_2$, while player 2 receives zero. The preemptive equilibrium neither payoff-dominates nor is dominated by the mixed-strategy equilibria.

B.1.4 Existence of a mixed-strategy equilibrium when no reinforcement equilibrium exists

It is worth noting that these mixed-strategy equilibria exist for the range of s where no reinforcement equilibrium exists, i.e., for $s > \bar{s}_1$ (see Proposition 1 in Clark et al. (2025)). To see this, we rewrite (B.24), denoting $\frac{V_1 - V_2}{s}$ by a and considering $\delta_1 < V_2/s$, as:

$$\begin{aligned} &\frac{(a+\delta_1)^2}{2\delta_1 s} + \frac{\delta_1}{s} + \frac{a}{s} < 1 \\ \iff\ & a^2 + 4a\delta_1 + 3\delta_1^2 < 2\delta_1 s \\ \iff\ & (a + 2\delta_1)^2 < 2\delta_1 s + \delta_1^2 \end{aligned} \tag{B.25}$$

Recall that $\bar{s}_1 = \dfrac{V_1 + V_2}{\sqrt{2V_2}}$, which implies that $s > \bar{s}_1 \iff \left(\dfrac{V_1 + V_2}{s}\right)^2 < 2V_2$. We can choose δ_1 less than but sufficiently close to V_2/s such that $2\delta_1 s < 2V_2 < 2\delta_1 s + \delta_1^2$. For such a combination of values of δ_1 and s, we then have:

$$(a + 2\delta_1)^2 < \left(\frac{V_1 - V_2}{s} + \frac{2V_2}{s}\right)^2 = \left(\frac{V_1 + V_2}{s}\right)^2 < 2V_2 < 2\delta_1 s + \delta_1^2;$$

Condition (B.25) is therefore satisfied.

B.1.5 An example of a mixed-strategy equilibrium

We construct a mixed-strategy Nash equilibrium for $s = \bar{s}_1$ following an approach that works for any $s > \bar{s}_1$.

Consider $\delta_1 = V_2/s$, so that $\alpha_1(0) = \dfrac{V_1^2}{2V_2 s^2}$, and $\alpha_2(0) = \dfrac{V_2}{2s^2}$. The condition (B.24) for the existence of a mixed-strategy equilibrium requires that $G_1(\delta_1) < 1$, or, equivalently, that $\dfrac{V_1^2 + 2V_1V_2}{2V_2} < s^2$. We consider $s = \bar{s}_1 = \dfrac{V_1 + V_2}{\sqrt{2V_2}}$, which satisfies (B.24) (note that all $s > \bar{s}_1$ satisfy the condition, as well). For $s = \bar{s}_1$, using (B.22), we get $\delta_2 = \dfrac{3V_2}{2\bar{s}_1}$.

Therefore, in this mixed strategy equilibrium, player 1 uses a strategy that puts a probability mass of $\alpha_1(0) = \dfrac{V_1^2}{(V_1 + V_2)^2}$ at zero, a probability mass of $\dfrac{\delta_1}{\bar{s}_1} + \dfrac{V_1 - V_2}{\bar{s}_1^2} = \dfrac{2V_1V_2}{(V_1 + V_2)^2}$ at $\dfrac{V_2}{\bar{s}_1}$, and randomization over $k \in \left(\dfrac{V_2}{\bar{s}_1}, \dfrac{3V_2}{2\bar{s}_1}\right]$ with a density $\dfrac{1}{\bar{s}_1} = \dfrac{\sqrt{2V_2}}{V_1 + V_2}$; while player 2 uses a strategy that puts a probability mass of $\alpha_2(0) = \dfrac{V_2^2}{(V_1 + V_2)^2}$ at zero, a probability mass of $\dfrac{\delta_1}{\bar{s}_1} = \dfrac{2V_2^2}{(V_1 + V_2)^2}$ at $\dfrac{V_1}{\bar{s}_1}$, randomization over $k \in \left(\dfrac{V_1}{\bar{s}_1}, \dfrac{V_1}{\bar{s}_1} + \dfrac{V_2}{2\bar{s}_1}\right)$ with a density $\dfrac{1}{\bar{s}_1} = \dfrac{\sqrt{2V_2}}{V_1 + V_2}$, and a probability mass of $1 - \dfrac{4V_2^2}{(V_1 + V_2)^2}$ at $\dfrac{V_1}{\bar{s}_1} + \dfrac{V_2}{2\bar{s}_1}$.

B.2 Weak perfection

Simon and Stinchcombe (1995) (hereafter SS) provide a generalization of the concept of perfect equilibrium—originally introduced by Selten (1975) for finite games—to infinite games with compact, metrizable strategy sets and continuous payoffs. In this section, we examine an SS-type perfect-equilibrium refinement of pure strategy equilibria of the first-period investment game. To invoke SS's framework, we restrict each player's action space to a closed interval. Specifically, consider an arbitrarily large but finite $K > \sqrt{2V_1}$ and confine a player's action choice k_i to $[0, K]$. This truncation does not alter first-period equilibrium behavior, since any $k_i > K$ is strictly dominated for both players.

A mixed strategy over the action space $[0, K]$ can be viewed as a probability measure on $[0, K]$. To determine the distance between two mixed strategies, we equip the strategy space with the Lévy–Prokhorov metric. Formally, if f and g are two probability measures on $[0, K]$, the distance between them is given by

$$\tau(f, g) = \inf\{\epsilon > 0 : \forall B \subset [0, K], f(B) \leq g(B^\epsilon) + \epsilon \text{ and } g(B) \leq f(B^\epsilon) + \epsilon\},$$

where B^ϵ is the ϵ-neighborhood of B. This metric induces the topology of weak convergence on the space of mixed strategies.

For $i \in \{1,2\}$, let Δ_i denote player i's set of mixed strategies over the action space $[0,K]$; and let $\Delta = \Delta_1 \times \Delta_2$. Further, for $\mu \in \Delta$, let $Br_i(\mu)$ denote i's set of best responses to the strategy profile μ.

Definition B.1. *[Simon and Stinchcombe (1995), Definition 1.2] A weakly ϵ-perfect equilibrium is a strategy profile $\mu^\epsilon = (\mu_1^\epsilon, \mu_2^\epsilon) \in \Delta$ such that, for $i \in \{1,2\}$, $\tau(\mu_i^\epsilon, Br_i(\mu^\epsilon)) < \epsilon$. A strategy profile $\mu = (\mu_1, \mu_2) \in \Delta$ is a weakly perfect equilibrium if it is the weak limit (converging point-wise at every point of continuity) of a weak ϵ_n-perfect equilibrium for some sequence $\epsilon_n \to 0$.*

The following theorem provides an equivalent characterization of weakly perfect equilibrium.

Theorem B.1. *[Simon and Stinchcombe (1995), Theorem 2.5] For the first-period investment game, the following statements are equivalent:*

(a) μ is a weakly perfect equilibrium; and

(b) μ is the limit of a sequence μ^n of full support strategies with the property that, for all $i \in \{1,2\}$, $\mu_i^n(Br_i(\mu^n)^{\epsilon_n}) \to 1$ for some sequence $\epsilon_n \to 0$.

Note that the two pure-strategy equilibria of the original game—the reinforcement equilibrium and the preemptive equilibrium—are also equilibria of the game with truncated action space. We further claim that these equilibria are weakly perfect whenever they are strict, i.e., whenever each player's strategy is the unique best response to the other player's strategy. Note that the reinforcement equilibrium is a strict equilibrium for $s \in (0, \bar{s}_1)$ and the preemptive equilibrium is a strict equilibrium for $s \in (\underline{s}_2, \bar{s}_2)$.

Let us first consider the preemptive equilibrium strategy profile $\underline{k}^*$ where $k_1^* = 0$ and $k_2^* \in \{z(s), \frac{V_1}{s}\}$. The following lemma shows that we can construct a sequence of full support strategy profiles $\{\mu^n\}$—such that their best responses lie sufficiently close to the pure strategy equilibrium—converging to the pure-strategy equilibrium such that, for each n, the best response to μ^n lies arbitrarily close to μ^n itself.

Lemma B.9. *For $s \in (\underline{s}_2, \hat{s}_2)$, the preemptive equilibrium $(k_1^* = 0, k_2^* = z(s))$ is a weakly perfect equilibrium.*

Proof. Let $(\delta_1, \delta_2) \in (0,1) \times (0,1)$ and define a strategy profile $\mu = (\mu_1, \mu_2)$ such that μ_i assigns probability $(1 - \delta_i)$ to k_i^* and distributes the remaining mass δ_i

uniformly over $[0, K]$. Given μ_1, player 2's expected payoff is:

$$\begin{aligned} EU_2(k_2 \mid \mu_1) = (1-\delta_1)s\Bigg[\max\{k_2 - \tfrac{V_1}{s} + \tfrac{V_2}{s}, 0\} - \max\{k_2 - \tfrac{V_1}{s}, 0\} \Bigg] \\ + \frac{\delta_1}{K}\Bigg[\int_0^{\max\{k_2 - \frac{V_1}{s}, 0\}} V_2 dk_1 \\ + \int_{\max\{k_2 - \frac{V_1}{s}, 0\}}^{\max\{k_2 - \frac{V_1}{s} + \frac{V_2}{s}, 0\}} ((V_2 - V_1) + s(k_2 - k_1)) dk_1 \Bigg] - c(k_2). \quad \text{(B.26)} \end{aligned}$$

It is obvious that for $k_2 < \frac{V_1 - V_2}{s}$, player 2's payoff is strictly decreasing in k_2. For $k_2 > \frac{V_1}{s}$, the term in the first square bracket is constant, whereas the term in the second square bracket is increasing in k_2. Applying (B.6) after substituting G_1 for a uniform distribution over $[0, K]$, the first-order derivative of player 2's expected payoff with respect to k_2 can be expressed as $\frac{\delta_1 V_2}{K} - c'(k_2)$, which is negative for all δ_1 sufficiently close to zero. For $k_2 \in [\frac{V_1 - V_2}{s}, \frac{V_1}{s}]$, the expected payoff simplifies to

$$EU_2(k_2 \mid \mu_1) = (1-\delta_1)(sk_2 + V_2 - V_1) + \frac{\delta_1 (sk_2 + V_2 - V_1)^2}{2sK} - c(k_2). \quad \text{(B.27)}$$

We now show that for $\underline{s}_2 < s < \hat{s}_2$, player 2's best response lies arbitrarily close to $z(s)$ as δ_1 approaches zero. Observe that for $k_2 \in [\frac{V_1 - V_2}{s}, \frac{V_1}{s}]$, $EU_2(k_2 \mid \mu_1)$ is strictly concave and so player 2's best response uniquely solves the following first-order condition:

$$F_2(k_2, \delta_1) := \Big[s - c'(k_2) \Big] + \delta_1 \Bigg[\frac{sk_2 + V_2 - V_1}{K} - s \Bigg] = 0. \quad \text{(B.28)}$$

Observe that $F_2(k_2, \delta_1)$ is continuous in k_2 and $z(s)$ solves $F_2(k_2, 0) = 0$. Therefore, for every $\epsilon > 0$, there exists $\delta_1 = \delta_1(\epsilon) > 0$ such that the unique solution of $F_2(k_2, \delta_1) = 0$ lies within ϵ-distance of $z(s)$, and moreover, $\delta_1(\epsilon) \to 0$ as $\epsilon \to 0$. Additionally, since the left-hand side (B.28) measures the first-order effect only for $k_2 \in [\frac{V_1 - V_2}{s}, \frac{V_1}{s}]$, the solution is indeed the best response if it lies in $[\frac{V_1 - V_2}{s}, \frac{V_1}{s}]$. However, since $z(s) < \frac{V_1}{s}$ for $\underline{s}_2 < s < \hat{s}_2$, we can choose ϵ sufficiently small so that for such ϵ, the solution to (B.28) is strictly below $\frac{V_1}{s}$ and thus constitutes player 2's local best response to μ_1.

Finally, this local solution constitutes a global best response for player 2 only if it yields a strictly positive payoff. Indeed, since the pure-strategy equilibrium is strict for $\underline{s}_2 < s < \hat{s}_2$, Proposition 2 gives

$$M(s) = sz(s) - c(z(s)) > V_1 - V_2.$$

Therefore, for sufficiently small $\epsilon > 0$, the expected payoff to player 2 from the solution of $F_2(k_2, \delta_1) = 0$ in response to μ_1 remains strictly positive.

Next, observe that given μ_2, player 1's expected payoff is:

$$\begin{aligned} EU_1(k_1 \mid \mu_2) = (1-\delta_2)s\Bigg[& \max\{k_1 - z(s) - \tfrac{V_2}{s} + \tfrac{V_1}{s}, 0\} - \max\{k_1 - z(s) - \tfrac{V_2}{s}, 0\}\Bigg] \\ & + \frac{\delta_2}{K}\Bigg[\int_0^{\max\{k_1 - \frac{V_2}{s}, 0\}} V_1 dk_2 \\ & + \int_{\max\{k_1 - \frac{V_2}{s}, 0\}}^{\max\{k_1 - \frac{V_2}{s} + \frac{V_1}{s}, 0\}} ((V_1 - V_2) + s(k_1 - k_2))dk_2\Bigg] - c(k_1). \end{aligned} \tag{B.29}$$

Consider $k_1 < z(s) - \frac{V_1 - V_2}{s}$, then for $\underline{s}_2 < s < \hat{s}_2$, $k_1 - \frac{V_2}{s} < z(s) - \frac{V_1}{s} < 0$ and $k_1 - \frac{V_2}{s} + \frac{V_2}{s}$ can be positive but less than $z(s)$. Therefore, the expected payoff simplifies to $\frac{\delta_2(sk_1 + V_1 - V_2)^2}{2sK} - c(k_1)$ and the first-order derivative is given by $F_1(k_1, \delta_2) := \frac{\delta_2(sk_1 + V_1 - V_2)}{K} - c'(k_1)$. Observe that $F_1(k_1, \delta_2)$ is continuous in k_1 and the solution of $F_1(k_1, \delta_2) = 0$ can be made arbitrarily close to zero by suitable choice of δ_2. Specifically, for every $\epsilon > 0$, there exists $\delta_2 = \delta_2(\epsilon) > 0$ such that the unique solution of $F_1(k_1, \delta_2) = 0$ is less than ϵ, and moreover, $\delta_2(\epsilon) \to 0$ as $\epsilon \to 0$.

For $k_1 > z(s) + \frac{V_2}{s}$, the term in the first square bracket is constant, whereas the term in the second square bracket is increasing in k_1. Applying (B.5) after substituting G_2 by a uniform distribution over $[0, K]$, the first-order derivative of player 1's expected payoff with respect to k_1 can be expressed as $\frac{\delta_2 V_1}{K} - c'(k_1)$, which is negative for all $k_1 > z(s) + \frac{V_2}{s}$ and for sufficiently small values of δ_2.

Finally, for $z(s) - \frac{V_1 - V_2}{s} \le k_1 \le z(s) + \frac{V_2}{s}$, the expected payoff simplifies to

$$EU_1(k_1 \mid \mu_2) = (1-\delta_2)(sk_1 - sz(s) + V_1 - V_2) + \frac{\delta_2(sk_1 + V_1 - V_2)^2}{2sK} - c(k_1),$$

which is strictly concave and so player 1's best response uniquely solves the following first-order condition:

$$\hat{F}_1(k_1, \delta_2) := \Big[s - c'(k_1)\Big] + \delta_2\left[\frac{sk_2 + V_2 - V_1}{K} - s\right] = 0. \tag{B.30}$$

Observe that $\hat{F}_1(k_1, \delta_2)$ is continuous in k_1 and $z(s)$ solves $\hat{F}_1(k_1, 0) = 0$. Therefore, for every $\epsilon > 0$, there exists $\delta_2 = \delta_2(\epsilon) > 0$ such that the unique solution of $\hat{F}_1(k_1, \delta_2) = 0$ lies within ϵ-distance of $z(s)$, and moreover, $\delta_2(\epsilon) \to 0$ as $\epsilon \to 0$. However, this local solution is not the global solution if it yields a negative payoff to player 1; then he deviates to the other local solution—which is arbitrarily close to zero—obtained in the first case $k_1 < z(s) - \frac{V_1 - V_2}{s}$. Indeed, since the pure-strategy equilibrium is strict for $\underline{s}_2 < s < \hat{s}_2$, Proposition 2 gives

$$c(z(s)) > V_1 - V_2,$$

and $EU_1(k_1 \mid \delta_2 = 0, \mu_2) = V_1 - V_2 - c(z(s)) < 0$. Therefore, for sufficiently small $\epsilon > 0$, the expected payoff to player 1 from the solution of $\hat{F}_1(k_1, \delta_2) = 0$ in response to μ_2 remains strictly negative.

Combining our analyses of the optimal behaviors of player 1 and player 2, we can construct a sequence of $\epsilon_n \downarrow 0$ and consider $\delta_n = \min\{\delta_1(\epsilon_n), \delta_2(\epsilon_n)\}$ such that the best response to a strategy profile $\{\mu^n = (\mu_1^n, \mu_2^n)\}$—where μ_i^n places a point mass $(1 - \delta_n)$ on k_i^* (where $k_1^* = 0$ and $k_2^* = z(s)$) and distributes the remaining probability δ_n uniformly over $[0, K]$—lies within ϵ_n-distance of μ^n.

Furthermore, as $\epsilon_n \to 0$, the sequence of strategy profile μ^n converges to $(k_1^* = 0, k_2^* = z(s))$. It then follows from Theorem B.1 that the pure strategy equilibrium $\underline{k}^*$ is weakly perfect. □

By analogous arguments, it can be shown that the preemptive equilibrium $(k_1^* = 0, k_2^* = \frac{V_1}{s})$ is weakly perfect also for $s \in [\hat{s}_2, \bar{s}_2)$, and the reinforcement equilibrium is weakly perfect for $s \in (0, \bar{s}_1)$, since in each case the corresponding pure-strategy equilibrium is strict. Strictness alone ensures compliance with the weak-perfection refinement, regardless of the presence of other equilibria.

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