Investment and endogenous efficiency in a contest

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Abstract

Contests are ubiquitous but do not happen in a vacuum. Rivals can prepare themselves for the contest to improve their ultimate chance of victory. Two contestants with different prize values play an all-pay auction and can invest to improve the efficiency of their own effort in the contest. We show that at most one player will invest, and that two asymmetric pure-strategy equilibria exist depending upon the identity of the investor. If the high-value player invests, then investment reinforces the initial asymmetry; investment by the low-value player turns the tables on the initially advantaged rival. The investment opportunity moves competition away from the contest, resulting in less expected contest effort than would occur without investment.

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1 Introduction

Contests are ubiquitous. Situations in which rivals expend irretrievable resources to win a prize are found generally in economics and politics. Contests have been used to model a wide range of applications such as innovation races, promotion battles, lobbying and armed conflict. Most research is concerned with the actual contest stage and the resources expended there to win the ultimate prize. However, important parts of actual competitions occur before the final showdown (Danhof, 1968; Lichtenberg, 1988). Knowing that a contest will occur gives rivals an opportunity to undertake actions designed to improve their chances of future success. Differences between rivals in the initial chance of winning is commonly modeled as an *ex-ante* exogenous bias in the contest success function.[1] The bias can also be set by a contest designer to achieve a specific objective such as contest effort maximization (Fu, 2006; Epstein et al., 2011; Li and Yu, 2012; Franke et al., 2018; Fu et al., 2024).

We consider an asymmetric all-pay auction between two rivals who differ in their valuation of the contest prize, analyzing their incentives to engage in pre-contest activities that determine the size and direction of any bias in the contest success function. We find two asymmetric equilibria with investments in the pre-contest stage, both sharing the feature that only one of the players invests. This stems from the very competitive nature of the contest stage, in which at least one player always expects a payoff of zero, making investment an activity with an expected loss for this player. Another feature of our equilibria is that investments lower contest efforts compared to the situation where pre-contest activities are not made.

In one of the equilibria in our model, called the reinforcement equilibrium, the high-valuation (strong) player invests to enhance the efficiency of her own effort in the contest, building upon her initial advantage. Importantly, we find conditions such that another equilibrium exists, called the preemptive equilibrium, in which the initially weaker player invests enough to become the stronger player in the contest. The contest is very competitive, and the initial prospects of the *ex-ante* weak player are rather dim. For her to invest, it must be in order to overtake her rival and enter the contest with an advantage. When the players' valuations do not differ too much, this strategy can pay off in equilibrium. Hence, conditions for the preemptive equilibrium to occur are stricter than for the reinforcement equilibrium, and the effect that investments has on contest effort is smaller. In the reinforcement equilibrium, all investment makes the rivals less equal, leading to less contest effort.

[1] See Mealem and Nitzan (2016) for a survey.

However, when the weaker player invests, as in the preemptive equilibrium, some of the investment goes to making the rivals more similar so that the heterogeneity between competitors is reduced, making for a more intense contest.

Whilst previous literature is quite limited, some models do incorporate how pre-contest investment affects the abilities of rivals to compete. In a Tullock contest, Fu and Lu (2009), Arbatskaya and Mialon (2012), and Schaller and Skaperdas (2020) assume that the effect of the investment is to improve the efficiency of contest effort, while Amegashie (2012) analyzes investments that increase rivals' cost in the contest. In an all-pay auction, Clark and Kundu (2024) consider probabilistic skill-enhancing investment prior to the contest stage. In an analysis complementary to the present one, Clark et al. (2024) examine incentives to invest in a head start in an all-pay auction. This can lead to situations where the prize is not actually contested; that is not the case in the current model with bias since here rivals will always exert effort in the contest. More closely related to the present work, Münster (2007) considers investments that affect the cost of making contest effort. Like us, he obtains asymmetric pure strategy equilibria where only one player invests ahead of an all-pay auction, also finding that this reduces fighting at the contest stage. In considering only *ex-ante* symmetric players, his model does not capture the phenomenon we highlight here, where the initially weak player can invest and become the stronger player in the contest. The reduction in contest effort in his model stems from the fact that the investment makes initially symmetric players asymmetric; this is well known to reduce contest effort.

Section 2 sets up the basic model which is then analyzed in Section 3. Section 4 discusses the results. All proofs are in the Appendix.

2 Model

Two risk neutral rivals play a two-period game. In period 1, each of them can make an investment in order to influence how effective their effort will be in the upcoming contest. Let $b_i \geq 0$ be the investment of player $i = 1, 2$ in period 1, with investment costs $C(b_i) = cb_i$ and $c > 0$. In period 2, the rivals play an all-pay auction with bias. The two players have prize valuations V_1 and V_2, such that $V_1 \geq V_2 > 0$.

The players' win probabilities in the period-2 contest depend on their efforts in period 2 as well as their investments in period 1. Player i has a score in this contest given by:

$$S_i = (1 + b_i)\, x_i,$$

where $(1 + b_i)$ measures a multiplicative bias in favor of contestant $i \in \{1, 2\}$, and

x_i is the player's effort in the period-2 contest. Contest effort has a unit cost for each player.

Player i has an expected (gross) payoff from taking part in the period-2 contest of

$$EU_i \ = \rho_i V_i - x_i,$$

where ρ_i is that player's probability of winning the contest given by:

$$\rho_i(S_1, S_2) = \begin{cases} 1, & \text{if } S_i > S_j; \\ \frac{1}{2}, & \text{if } S_i = S_i; \text{and} \\ 0, & \text{if } S_i < S_j. \end{cases}$$

The probability of player j winning is $1 - \rho_i(S_1, S_2)$.

A player's expected net payoff in this game is:

$$EW_i(b_i, b_j) = EU_i - cb_i.$$

We characterize the pure-strategy subgame-perfect Nash equilibria of this two-period game.

3 Analysis

3.1 The second-period contest

In the all-pay auction, player i is strong if $(1 + b_i)V_i \geq (1 + b_j)V_j$. Lemma 1, which is well known,[2] shows that only the strong player can come out of the contest with a positive expected payoff.

Lemma 1.

If $(1 + b_i)V_i \geq (1 + b_j)V_j$, then the unique mixed-strategy equilibrium in the contest gives expected efforts:

$$Ex_i(b_i, b_j) = \frac{(1 + b_j)V_j}{2(1 + b_i)}; \quad Ex_j(b_i, b_j) = \frac{(1 + b_j){V_j}^2}{2(1 + b_i)V_i};$$
$$EX(b_i, b_j) := Ex_i(b_i, b_j) + Ex_j(b_i, b_j) = \frac{(1 + b_j)V_j}{(1 + b_i)V_i} \left(\frac{V_i + V_j}{2} \right) \tag{1}$$

[2]See for example Fu (2006) and Clark and Nilssen (2020).

and expected payoffs:

$$EU_i = V_i - \frac{1+b_j}{1+b_i} V_j; \quad EU_j = 0.$$

Note that the stronger player i is, the less total expected effort is expended in the contest, and the larger is the expected payoff of that player. When $(1+b_i)V_i = (1+b_j)V_j$, the contest is balanced: fighting is intense and both players expect a payoff of zero.

3.2 The first-period investment

We focus on pure-strategy equilibria in the first period. Lemma 1 shows that at most one player expects a positive payoff in the all-pay auction. Consequently, at most one player will use a pure strategy involving positive investment.[3] We have:

Proposition 1. *There is no pure-strategy equilibrium in which both players simultaneously make non-zero investments.*

Using Lemma 1, we can express the players' expected first-period payoffs as functions of their investments levels:

$$\begin{aligned} EW_1(b_1, b_2) &= \max\left\{V_1 - \frac{1+b_2}{1+b_1} V_2, 0\right\} - c \cdot b_1 \\ EW_2(b_1, b_2) &= \max\left\{V_2 - \frac{1+b_1}{1+b_2} V_1, 0\right\} - c \cdot b_2 \end{aligned}$$

Lemma 2 characterizes each player's best response:

Lemma 2. *Fix $b_2 \geq 0$. The best response of player 1 is given by*

$$BR_1(b_2) = \begin{cases} \max\left\{\sqrt{\frac{(1+b_2)V_2}{c}} - 1, 0\right\}, & \text{if } b_2 < \frac{(V_1 + \min\{V_1, c\})^2}{4cV_2} - 1; \\ 0, & \text{if } b_2 \geq \frac{(V_1 + \min\{V_1, c\})^2}{4cV_2} - 1. \end{cases} \tag{2}$$

Fix $b_1 \geq 0$. The best response of player 2 is given by

$$BR_2(b_1) = \begin{cases} \max\left\{\sqrt{\frac{(1+b_1)V_1}{c}} - 1, 0\right\}, & \text{if } b_1 < \frac{(V_2 + \min\{V_2, c\})^2}{4cV_1} - 1; \\ 0, & \text{if } b_1 \geq \frac{(V_2 + \min\{V_2, c\})^2}{4cV_1} - 1. \end{cases} \tag{3}$$

[3]The same argument appears in Münster (2007).

When the cost of investment is large, neither player invests. We refer to this as the status-quo equilibrium, since it is the one that would obtain without the investment opportunity.

Proposition 2. *For $c \geq V_2$, the unique pure strategy equilibrium at the investment stage is the status-quo equilibrium where $b_1 = b_2 = 0$. Expected payoffs are $EW_1(0,0) = V_1 - V_2$, $EW_2(0,0) = 0$.*

There are two distinct equilibria with positive investment. In the reinforcement equilibrium (Proposition 3), the high-valuation player invests to reinforce her advantage, while the low-valuation player makes no investment. In the preemptive equilibrium (Proposition 4), the low-valuation player makes an investment sufficiently large as to overturn the valuation-based advantage of the rival.

Proposition 3. *There exists a unique $\underline{c}_1 \in (0, V_2/3)$ such that, for every $c \in [\underline{c}_1, V_2)$, a reinforcement equilibrium exists in which player 1 invests $b_1^* = \sqrt{V_2/c} - 1 > 0$, earning an expected net payoff of $EW_1(b_1^*, 0) = (V_1 - V_2) + (\sqrt{V_2} - \sqrt{c})^2 > 0$. Player 2 invests nothing and has an expected net payoff of $EW_2(b_1^*, 0) = 0$.*

The case of symmetry ($V_1 = V_2 = V$) is an immediate consequence of Propositions 2 and 3:

Corollary 1. *Suppose $V_1 = V_2 = V$. a) For $c \geq V$, the unique pure strategy equilibrium at the investment stage is $b_1 = b_2 = 0$ with expected payoffs $EW_1(0,0) = EW_2(0,0) = 0$. b) For $c \in [0.087V, V)$, player i invests $b_i^* = \sqrt{V/c} - 1 > 0$ and player j invests $b_j^* = 0$, $i, j = 1, 2, i \neq j$. Expected net payoffs are $EW_i(b_i^*, 0) = (\sqrt{V} - \sqrt{c})^2 > 0$, and $EW_j(b_i^*, 0) = 0$.*

This is qualitatively similar to the findings of Münster (2007) who considers only symmetric players. Two asymmetric pure strategy equilibria with investment exist as long as the investment cost is above a minimum level.[4] When players are not assumed to be *ex-ante* identical, an important case arises in which the weak player invests in order to gain the upper hand in the contest.

Proposition 4. *Let $V_2/V_1 > \bar{v}$, where $\bar{v} \in (4\sqrt{3}/9, 1)$. There exist unique thresholds $\underline{c}_2 \in (0, V_1/3)$ and $\bar{c}_2 \in (0, V_2]$, with $\underline{c}_1 \leq \underline{c}_2 < \bar{c}_2$, such that, for every $c \in [\underline{c}_2, \bar{c}_2)$, a preemptive equilibrium exists in which player 2 invests $b_2^* = \sqrt{V_1/c} - 1 > 0$, with an expected net payoff of $EW_2(0, b_2^*) = (V_2 - V_1) + (\sqrt{V_1} - \sqrt{c})^2 \geq 0$, with strict inequality for $c \in (\underline{c}_2, \bar{c}_2)$. Player 1 invests nothing and has an expected payoff of $EW_1(0, b_2^*) = 0$.*

[4] This is to ensure that the non-investing player will not deviate. See Section 4.

Several requirements must be met for the preemptive equilibrium in Proposition 4 to exist. First, the initial valuations must be sufficiently close, and second, the cost cannot be too low or too high. When V_2 is sufficiently close to V_1, two things are ensured: first, that the initially weaker player can become strong through the equilibrium investment level, and second, that player 1 does not deviate from the non-investment strategy. The temptation for this player to deviate is increasing in the difference between the valuations, as can be seen from the expected payoff to the strong player in Lemma 1. When the cost of investing is too high, it is not profitable for the weak player to invest; when the cost of investing is too low, the preemptive equilibrium breaks down, as the strong player will not stick to the equilibrium strategy of non-investment.

Note that, since $\underline{c}_1 < \underline{c}_2 < \bar{c}_2 < V_2$, the preemptive equilibrium exists only when also the reinforcement equilibrium exists. Figure 1 illustrates existence of the three equilibria for $V_1 = 1$, making this also the upper limit for V_2. The preemptive equilibrium only occurs for a sufficiently high V_2/V_1.

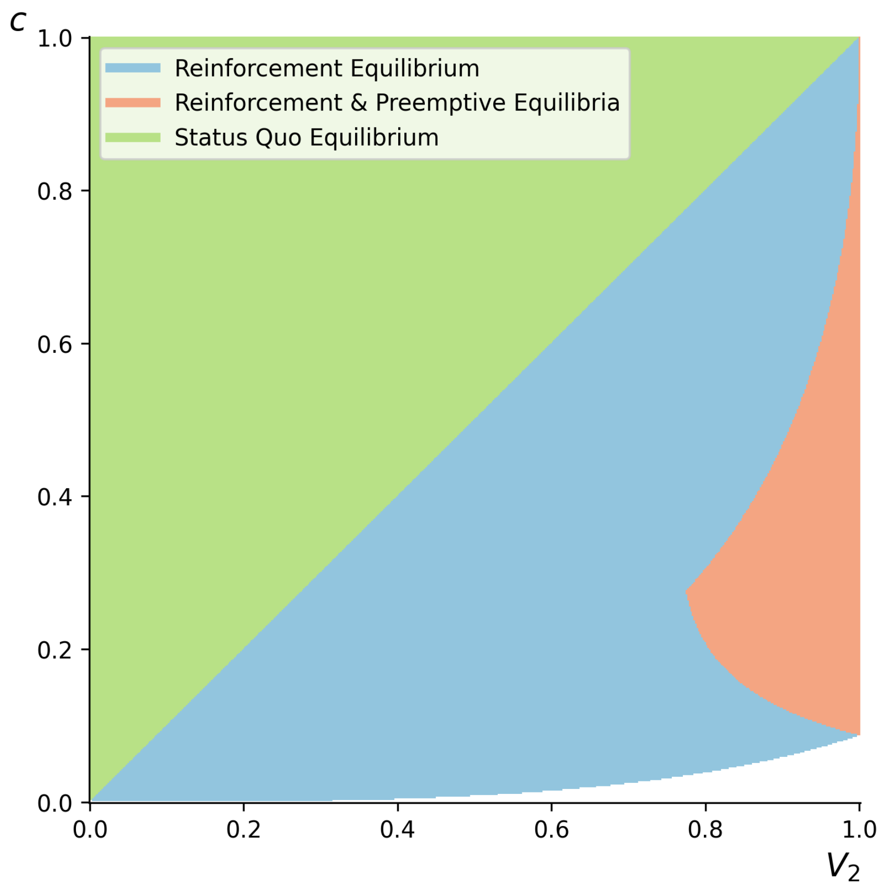

Figure 1: Existence of equilibria: $V_1 = 1$

Proposition 5 derives total expected effort in the three equilibria. The comparison between expected effort in the reinforcement and preemptive equilibria is valid for a common c; in the proof, we establish that this is $c \in [\underline{c}_2, \bar{c}_2)$.

Proposition 5. *Let $\theta := \frac{V_1+V_2}{2}$. Total expected effort is given by*

$$EX(0,0) = \frac{V_2}{V_1}\theta; \quad EX(b_1^*,0) = \frac{\sqrt{cV_2}}{V_1}\theta; \quad EX(0,b_2^*) = \frac{\sqrt{cV_1}}{V_2}\theta.$$

Furthermore, $EX(0,0) > EX(0,b_2^*) > EX(b_1^*,0)$, where the latter comparison is valid for $c \in [\underline{c}_2, \bar{c}_2)$.

Figure 2 illustrates the results from Propositions 2 through 5. The investment schedules, total expected efforts, and expected payoffs are shown as whole lines for the reinforcement equilibrium, dashed lines for the preemptive equilibrium, and dot-dashed lines for the status quo equilibrium.

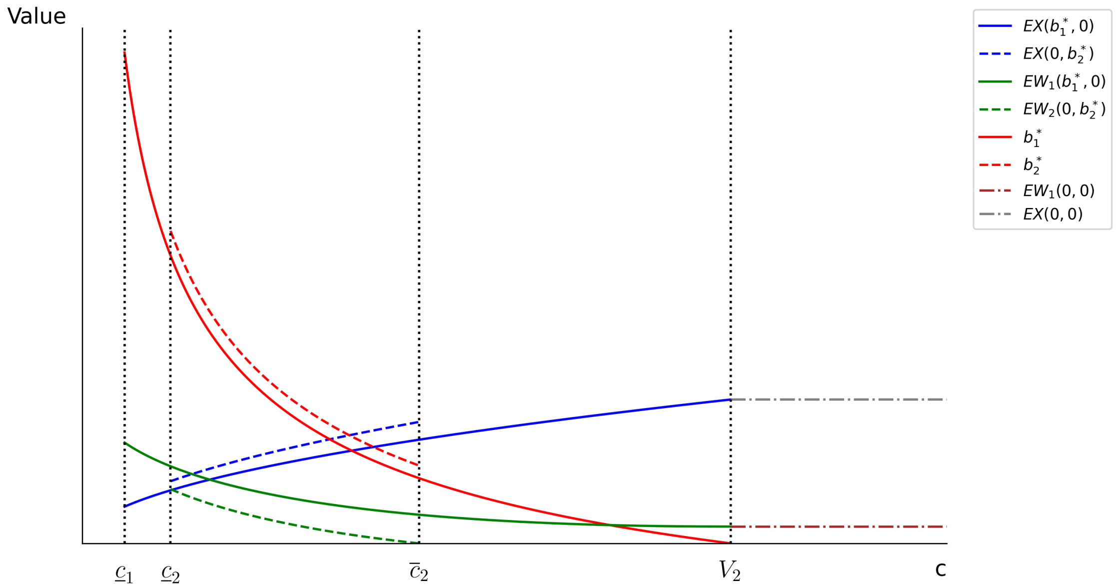

Figure 2: Comparison of equilibria: $V_1 = 1, V_2 = 0.9$

4 Discussion

In both investment equilibria, a lower bound for c is necessary to ensure that one of the players invests zero; as c gets very small, the cost of investing diminishes, making a deviation more likely for the non-investing player. As c increases, so does expected effort in both equilibria. In the reinforcement equilibrium, an increase in investment cost leads to less investment by player 1, lessening the difference between the players. More similar rivals fight more intensely at the contest stage.

The investment equilibrium continues to exist in this case until c reaches V_2. In the preemptive case, an increase in cost leads to less investment by player 2, lessening the difference between her and the now weaker rival. This leads to a more intensely fought contest.

Figure 2 makes it clear that the investing player expects a larger net payoff than in the status quo equilibrium in spite of the investment cost. In each investment equilibrium, total expected contest effort falls when the valuation of the investing player increases, as this exacerbates differences in their contest strength. Increasing the valuation of the non-investing player has two effects that work in opposite directions. The direct effect increases the contest strength of the weaker player, lessening the difference between the rivals which increases expected effort. The indirect effect works by increasing the investment of the opponent, making the players less equal and hence decreasing effort. It is straightforward to show that the former, direct effect dominates so that increasing the valuation of the non-investing player leads to a higher level of expected effort in the contest.

A Appendix

A.1 Proof of Lemma 2

Proof. Consider player 1's best response. Fix $b_2 \geq 0$. Define $\bar{b}_1 := (1+b_2)V_2/V_1 - 1$. Observe that $(1+b_1)V_1 \leq (1+b_2)V_2$ is equivalent to $b_1 \leq \bar{b}_1$.

For $b_1 \leq \bar{b}_1$, player 1's payoff is decreasing in b_1, so her optimal choice of investment is $b_1 = 0$.

For $b_1 > \bar{b}_1$, player 1's payoff is locally maximized (satisfying both the first-order and second-order conditions) at $\hat{b}_1 := \sqrt{(1+b_2)V_2/c} - 1$, conditional on $\hat{b}_1 > \bar{b}_1$ and $\hat{b}_1 \geq 0$.

If $\hat{b}_1 \leq \bar{b}_1$, which, after simplification, reduces to $b_2 \geq V_1^2/cV_2 - 1$, then player 1's payoff decreases for all $b_1 \geq 0$, which follows from (a) $dEW_1(b_1, b_2)/db_1 < 0$ for $b_1 < \bar{b}_1$ and for $b_1 > \bar{b}_1$, and (b) continuity of $EW_1(b_1, b_2)$ at $\bar{b}_1$. Therefore, player 1's best response is $b_1 = 0$.

If $\hat{b}_1 > \bar{b}_1$ and $\hat{b}_1 \geq 0$, then player 1 receives at $\hat{b}_1$ a locally maximum expected payoff of $V_1 + c - 2\sqrt{c(1+b_2)V_2}$, which is strictly positive if $b_2 < (V_1+c)^2/(4cV_2) - 1$.

These observations together imply that player 1 would choose a strictly positive investment level only if the two inequalities, $b_2 < (V_1+c)^2/(4cV_2) - 1$ and $b_2 < V_1^2/cV_2 - 1$, hold simultaneously, in which case her best response is $\hat{b}_1$, conditional on $\hat{b}_1 > 0$. Thus, we find that the best response of player 1 is given by (2).

Player 2's investment incentives are similar although the exact payoffs differ based on player-specific parameter values, and his best response function can be derived using arguments similar to those presented above. □

A.2 Proof of Proposition 2

Proof. Suppose $c \geq V_2$ and $b_2 = 0$. Equation (2) implies that $BR_1(0) = 0$. Given $b_1 = 0$, $BR_2(0) = 0$ if $b_1 \geq (V_2 + V_2)^2/4cV_1 - 1$, which reduces to $c \geq V_2^2/V_1$. This is true since $c > V_2 \geq V_2^2/V_1$, where the latter inequality holds since $V_1 \geq V_2$.

□

A.3 Proof of Proposition 3

Proof. Consider $b_2 = 0$. It follows from (2) that player 1's best response is strictly positive only if both of the following conditions hold: (a) $\sqrt{V_2/c} - 1 > 0$, and (b) $(V_1 + \min\{V_1, c\})^2/(4cV_2) > 1$.

Condition (a) implies $c < V_2$. Furthermore, if (a) is satisfied, then (b) reduces to $(V_1 + c)^2 > 4cV_2$, which always holds since $(V_1 + c)^2 - 4cV_2 \geq (V_2 + c)^2 - 4cV_2 = (V_2 - c)^2 > 0$. Therefore, $BR_1(0) = \sqrt{V_2/c} - 1 > 0$ if $c < V_2$.

Next, consider $c < V_2$. It follows from (3) that $BR_2(\sqrt{V_2/c} - 1) = 0$ only if $\sqrt{V_2/c} - 1 \geq (V_2 + \min\{V_2, c\})^2/4cV_1 - 1$, or equivalently, $l_1(c) := (V_2+c)^2/4V_1\sqrt{cV_2} - 1 \leq 0$. Note that $\lim_{c\to 0+} l_1(c) > 0$, $l_1(V_2) \leq 0$, and $l_1'(c) \lesseqgtr 0$ if $c \lesseqgtr V_2/3$. Therefore, there exists a unique $\underline{c}_1 \in (0, V_2/3)$ satisfying $l_1(c) = 0$ such that, for $c \in [\underline{c}_1, V_2]$, $l_1(c) \leq 0$ and, consequently, $BR_2(\sqrt{V_2/c} - 1) = 0$.

Player 1's expected payoff is $V_1 - V_2/\sqrt{V_2/c} - c(\sqrt{V_2/c} - 1) = V_1 + c - 2\sqrt{cV_2} = (V_1 - V_2) + (\sqrt{V_2} - \sqrt{c})^2$. Player 2 receives zero expected payoff. □

A.4 Proof of Proposition 4

Proof. Consider $b_1 = 0$. By Lemma 3, $BR_2(0) > 0$ if (a) $\sqrt{V_1/c} - 1 > 0$, and (b) $(V_2 + \min\{V_2, c\})^2/(4cV_1) - 1 > 0$. Assume $c < V_1$ (so that (a) is satisfied), and examine the following two ranges of values of c.

Case 1: $V_2 \leq c < V_1$. In this range, $(V_2 + \min\{V_2, c\})^2/(4cV_1) - 1 = V_2^2/cV_1 - 1 < 0$, and so (b) cannot be satisfied.

Case 2: $c < V_2$. In this range, $(V_2 + \min\{V_2, c\})^2/(4cV_1) - 1 = (V_2 + c)^2/(4cV_1) - 1 =: h_2(c)$. Observe that $\lim_{c\to 0+} h_2(c) > 0$, $h_2(V_2) = V_2/V_1 - 1 \leq 0$, and $h_2'(c) < 0$.

Therefore, there exists a unique $\bar{c}_2 \in (0, V_2)$ such that $\forall c \in (0, \bar{c}_2)$, $h_2(c) > 0$ and $\forall c \in (\bar{c}_2, V_2)$, $h_2(c) < 0$, implying $BR_2(0) > 0$ only if $c \in (0, \bar{c}_2)$. Solving the equation directly yields $\bar{c}_2 = 2V_1 - V_2 - 2\sqrt{V_1(V_1 - V_2)} = (\sqrt{V_1} - \sqrt{V_1 - V_2})^2$.

Next, we consider $c \in (0, \bar{c}_2)$, $b_2 = \sqrt{V_1/c} - 1$ and examine when $BR_1(b_1)$ equals zero.

By Lemma 2, $BR_1(b_2) = 0$ if either (A) $b_2 \geq (V_1 + \min\{V_1, c\})^2/(4cV_2) - 1$, or (B) $(1 + b_2)V_2 < c$. For $c < \bar{c}_2$, (B) can never be satisfied. Further, (A) reduces to $\sqrt{V_1/c} \geq (V_1 + c)^2/(4cV_2)$, or, equivalently, $l_2(c) := (V_1 + c)^2/(4V_2\sqrt{cV_1}) - 1 \leq 0$.

Observe that $\lim_{c \to 0+} l_2(c) > 0$, and $l_2'(c) \lesseqgtr 0$ if $c \lesseqgtr V_1/3$. So a necessary condition for having $l_2(c) < 0$ for $c > 0$ is that $l_2(V_1/3) < 0 \Leftrightarrow V_2/V_1 > 4\sqrt{3}/9$.

Therefore, if $V_2/V_1 \leq 4\sqrt{3}/9$, then $l_2(c) \geq 0$ for all $c \geq 0$ and there exists no preemptive equilibrium.

If $V_2/V_1 > 4\sqrt{3}/9$, there will be two roots of $l_2(c) = 0$ around $V_1/3$, denoted by $\underline{c}_2$ and $\hat{c}_2$, such that $\underline{c}_2 < V_1/3 < \hat{c}_2$ and $\forall c \in [\underline{c}_2, \hat{c}_2]$, $l_2(c) \leq 0$.

Denote V_2/V_1 by v.

Claim 1: Consider $v > 4\sqrt{3}/9$. $\exists \bar{v} \in (4\sqrt{3}/9, 1)$ such that for $v \gtreqless \bar{v}$, $\underline{c}_2 \lesseqgtr \bar{c}_2$.

Proof of Claim 1: As $v > 4\sqrt{3}/9$, $l_2(c) \leq 0$ for $c \in [\underline{c}_2, \hat{c}_2]$, and $l_2(c) > 0$ otherwise. We complete the proof of Claim 1 in two steps.

In step 1, we will show that there exists a threshold $\bar{v} > 4\sqrt{3}/9$ such that for $v > \bar{v}$, $l_2(\bar{c}_2)$ is negative, implying that $\underline{c}_2 < \bar{c}_2 < \hat{c}_2$. In step 2, we will show that for $v \in (4\sqrt{3}/9, \bar{v}]$, $\bar{c}_2 \leq \underline{c}_2$.

Observe that

$$l_2(\bar{c}_2) = \frac{\left(3V_1 - V_2 - 2\sqrt{V_1(V_1 - V_2)}\right)^2}{4V_1V_2 - 4V_2\sqrt{V_1(V_1 - V_2)}} - 1 = \frac{\left(3 - v - 2\sqrt{1 - v}\right)^2}{4v - 4v\sqrt{1 - v}} - 1.$$

It follows that $l_2(\bar{c}_2) < 0 \Leftrightarrow (13 - v)(1 - v) + 8v\sqrt{1 - v} - 12\sqrt{1 - v} < 0$. Replacing $\sqrt{1 - v}$ by x, we can express the above inequality as $f(x) = x^4 - 8x^3 + 12x^2 - 4x < 0$ where $x = \sqrt{1 - v} \in (0, 1)$. Observe that $df/dx = 4(x^3 - 6x^2 + 6x - 1)$. Direct examination of df/dx shows that (i) it is negative at $x = 0$, (ii) it equals zero at $x = 1$, and (iii) it is a concave function (because $d^3f/dx^3 < 0$). For $x \in (0, 1)$, direct calculation gives a unique root at $x = (5 - \sqrt{21})/2$. This implies that there must be a threshold value of x, denoted by $\bar{x}$ such that $f(x) \lesseqgtr 0$ for $x \lesseqgtr \bar{x}$. Define $\bar{v} = 1 - \bar{x}^2$. Then, it follows that for $v > \bar{v}$, $f(x) < 0$, or equivalently, $l_2(\bar{c}_2) < 0$, implying that $\underline{c}_2 < \bar{c}_2 < \hat{c}_2$. It trivially follows that $\bar{v} > 4\sqrt{3}/9$, since only for $v > 4\sqrt{3}/9$, $l_2(c)$ can take negative values. Although we could not analytically solve $\bar{v}$, numerical algorithms show that $\bar{v} = 0.775$. This completes step 1 of the proof.

In step 2, we consider $v \in (4\sqrt{3}/9, \bar{v}]$; In this range, we already know that

$l_2(\bar{c}_2) \geq 0$. We claim that $\bar{c}_2$ must be lower than $V_1/3$, implying that $\bar{c}_2 \not\geq \hat{c}_2$, and so $\bar{c}_2 \leq \underline{c}_2$. Direct comparison shows $\bar{c}_2 \leq V_1/3 \Leftrightarrow (5 - 3v - 6\sqrt{1-v}) \leq 0$, which is equivalent to $v \leq (2\sqrt{3}-1)/3 \approx 0.82$. This upper limit of v is higher than $\bar{v}$, implying that for $v \in (4\sqrt{3}/9, \bar{v}]$, $\bar{c}_2 \leq \underline{c}_2$. This completes the proof of Claim 1.

Note finally that $\underline{c}_s$, $s = 1, 2$, solves $l_s(c) = 0$, where $l_1(c)$ is defined in the proof of Proposition 3. Since $l_2(c) > l_1(c)$, it follows that $\underline{c}_2 > \underline{c}_1$. This completes the proof of the proposition. □

A.5 Proof of Proposition 5

Proof. Expressions for total effort are obtained by simple substitution in (1). The comparison of $EX(0,0)$ with $EX(b_1^*, 0)$ is trivial. We have $EX(0,0) > EX(0, b_2^*)$ for $V_2/V_1 > \sqrt{cV_1}/V_2$, or equivalently $V_2^2/\sqrt{V_1}V_1 > \sqrt{c}$. If this holds for $c = \bar{c}_2$ then it will hold for all permitted c. Inserting into the inequality and rearranging gives $0 > V_1^2 - V_2^2 - V_1V_2$, which holds for $V_1 \in (V_2, (\sqrt{5}+1)/2 \cdot V_2)$. In the proof of Proposition 4, we show that the equilibrium is valid for $V_1 \in (V_2, 9/(4\sqrt{3} \cdot V_2))$. Since $(\sqrt{5}+1)/2 > 9/(4\sqrt{3})$, the inequality holds, and $EX(0,0) > EX(0, b_2^*)$.

The reinforcement equilibrium holds for $c \in [\underline{c}_1, V_2)$ and the preemptive one for $c \in [\underline{c}_2, \bar{c}_2)$. From Proposition 4, we have that $\underline{c}_1 < \underline{c}_2 < \bar{c}_2 < V_1/3 < V_2$, where the latter inequality follows from the existence condition $V_1/V_2 < 9/(4\sqrt{3})$. Hence the interval $[\underline{c}_2, \bar{c}_2)$ is completely contained in $[\underline{c}_1, V_2)$. For common $c \in [\underline{c}_2, \bar{c}_2)$, the comparison between $EX(b_1^*, 0)$ and $EX(0, b_2^*)$ is trivial.

□

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