Appendix C: Pooling equilibria

In this note, we study the pooling equilibria of the signaling subgame.

Lemma C1 *Suppose $x > 0$ and Assumption 1 holds. Then there exists a set of pooling equilibria in which both types take action a^* with $a^* \in [0, \frac{(1-\lambda)p\triangle v}{(1-\lambda)p\triangle v+\lambda(v^A+\underline{v})}]$ and $\triangle v = \overline{v} - \underline{v}$.*

Proof of Lemma C1. First we derive the optimal transfer for any belief $\mu \in [0,1]$ where μ denotes the probability of type $\underline{v}$.

Claim C1a: Suppose $x > 0$. For beliefs $\mu \in [0,1]$,

$$t(\mu, a) = \lambda w^A(a) - (1-\lambda)\left[\mu w^B(\underline{v}, a) + (1-\mu) w^B(\overline{v}, a)\right]$$

The transfer to group B is strictly increasing in μ if $a < 1$ and constant if $a = 1$.

Proof of Claim C1a: See the proof of Claim $L2a$ inside the proof of Lemma 2.

Next let us suppose, if possible, there exist a pooling equilibrium in which both types take action a^* in equilibrium. Consider the following belief $\mu = \left\{\begin{smallmatrix} p \text{ if } a=a^* \\ 0 \text{ if } a\neq a^* \end{smallmatrix}\right.$. At the equilibrium path, no information is revealed, and therefore, we must have $\mu = p$. Otherwise, we assume that B will be treated most unfavorably, so that $\mu = 0$. Such a belief is going to generate the maximum possible set of pooling equilibria, if it exists.

Let $w^B(a, t|v)$ denote group B's payoff given its true marginal valuation v, a redistributive transfer t, and an action a. It is easy to see if any of the types deviates to an action other than a^*, then her dominant action choice is $a = 0$. Thus the no-deviation conditions for type $\overline{v}$ and for type $\underline{v}$ are given by

$$w^B(0, t(0,0)|\overline{v}) \quad \leq \quad w^B(a^*, t(p,a^*)|\overline{v}) \tag{1}$$

$$w^B(0, t(0,0)|\underline{v}) \quad \leq \quad w^B(a^*, t(p,a^*)|\underline{v}) \tag{2}$$

By rearranging terms, we see that inequalities (1) and (2) can be summarized (respectively) as $\overline{v}a^*x \leq \Delta t(a)$ and $\underline{v}a^*x \leq \Delta t(a)$ where $\Delta t(a) = t(p, a^*) - t(0,0)$. In these inequalities, the left hand side represents the loss in valuation due to action, and the right hand represents the loss in transfer when a type is treated as the high valuation type with probability 1. As we will always have $\overline{v} > \underline{v}$, the two conditions will simultaneously be satisfied if

$$\overline{v}a^*x \leq \Delta t(a).$$

The loss in transfer $\Delta t(a)$ is given by $\Delta t(a) = x\left[(1-\lambda)p\triangle v(1-a^*) + a^*\left((1-\lambda)\overline{v} - \lambda v^A\right)\right]$. After rearranging terms, we see that in any pooling equilibrium, we must have

$$a^* \leq \frac{(1-\lambda)p\triangle v}{(1-\lambda)p\triangle v + \lambda(v^A + \underline{v})}. \tag{3}$$

Given Assumption 1, the right hand side of inequality (3) always lies in the interval $[0,1]$. Thus, we will have a set of pooling equilibria with $a^* \in [0, \frac{(1-\lambda)p\triangle v}{(1-\lambda)p\triangle v+\lambda(v^A+\underline{v})}]$, $t = t(\mu, a)$ and associated beliefs $\mu = \left\{\begin{smallmatrix} p \text{ if } a=a^* \\ 0 \text{ if } a\neq a^* \end{smallmatrix}\right.$.
■

The next lemma shows that no pooling equilibrium survives the intuitive criterion.

Lemma C2 *Suppose $x > 0$ and Assumption 1 holds. Then there is no pooling equilibrium that survives the intuitive criterion*

Proof of Lemma C2. Consider any specific pooling equilibrium with $a^* \in [0, \frac{(1-\lambda)p\triangle v}{(1-\lambda)p\triangle v+\lambda(v^A+\underline{v})}]$, $t = t(\mu, a)$ and associated beliefs $\mu = \left\{\begin{smallmatrix} p \text{ if } a=a^* \\ 0 \text{ if } a\neq a^* \end{smallmatrix}\right.$. Given such a pooling equilibrium, an action a is equilibrium dominated for the type $\overline{v}$ if

$$\max_{\mu\in[0,1]} w^B(a, t(\mu, a)|\overline{v}) < w^B(a^*, t(p, a^*)|\overline{v}).$$

As $t(\mu, a)$ is increasing in μ, we conclude that an action a is equilibrium dominated for the type $\overline{v}$ if

$$w^B(a, t(1, a)|\overline{v}) \leq w^B(a^*, t(p, a^*)|\overline{v}). \tag{4}$$

Claim C2a: There exists $B(a^*) \in (a^*, 1]$ such that the inequality (4) is equivalent to the condition $a \geq B(a^*)$.

Proof of Claim C2a: The claim will follow if we can show that $w^B(a, t(1, a)|\overline{v})$ is decreasing in a and $w^B(a^*, t(p, a^*)|\overline{v}) \leq w^B(a^*, t(1, a^*)|\overline{v})$. To see this, note that

$$\begin{aligned} w^B(a, t(1, a)|\overline{v}) &= \overline{v}x(1-a) + s\tau x + t(1, a) \\ &= \overline{v}x(1-a) + s\tau x + \left[\lambda\{v^A x(1-a) + (1-s)\tau x\} - (1-\lambda)\{\underline{v}x(1-a) + s\tau x\}\right] \\ &= x(1-a)\left[\triangle v + \lambda(v^A + \underline{v})\right] + \lambda\tau x, \end{aligned}$$

which is decreasing in a, given Assumption 1. And, we have $w^B(a^*, t(p, a^*)|\overline{v}) \leq w^B(a^*, t(1, a^*)|\overline{v})$ as $t(p, a^*) \leq t(1, a^*)$ and $w^B(a^*, t|\overline{v})$ is increasing in t. Hence, Claim C2a is proved.

We can derive $B(a^*)$ from the equation

$$w^B(B(a^*), t(1, B(a^*))|\overline{v}) = w^B(a^*, t(p, a^*)|\overline{v}). \tag{5}$$

Simplifying (5), we see that $B(a^*)$ satisfies

$$t(1, B(a^*)) - t(p, a^*) = \overline{v}(B(a^*) - a^*). \tag{6}$$

Therefore, if we see an action $a \geq B(a^*)$, under intuitive criterion, we can impose a reasonable belief that $\mu = 1$. Given such a belief, the payoff of type $\underline{v}$, if it takes an action $a = B(a^*)$, is given by $w^B(B(a^*), t(1, B(a^*))|\underline{v})$. The type $\underline{v}$ will deviate from the action a^* if

$$\begin{aligned} & w^B(B(a^*), t(1, B(a^*))|\underline{v}) - w^B(a^*, t(p, a^*)|\underline{v}) > 0 \\ \iff & t(1, B(a^*)) - t(p, a^*) > \underline{v}(B(a^*) - a^*) \\ \iff & \overline{v}(B(a^*) - a^*) > \underline{v}(B(a^*) - a^*) \\ \iff & \triangle v(B(a^*) - a^*) > 0 \end{aligned}$$

which is always true as $a^* < B(a^*)$. Thus the pooling equilibrium with $a = a^*$ does not survive the intuitive criterion. Since a^* can take any arbitrary value in $[0, \frac{(1-\lambda)p\triangle v}{(1-\lambda)p\triangle v + \lambda(v^A + \underline{v})}]$, it implies that no pooling equilibrium survives the intuitive criterion. ■